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NOTES ON PRACTICAL ASTRONOMY AND GEODESY

BY

L. B. STEWART, D.T.S.

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These notes contain an outline of the course of lectures in Practical Astronomy and Geodesy, that, for over twenty-five years, has been given to the students of the Third Year in the department of Civil Engineering in the Faculty of Applied Science of the University of Toronto. They are designed to fulfil the requirements of candidates desirous of obtaining a commission as a Dominion or Ontario Land Surveyor, and at the same time to provide a course of study suited to the needs of the engineer who does not intend to devote himself specially to this class of work.

LOUIS B. STEWART.

1996

NOTES ON PRACTICAL ASTRONOMY AND GEODESY.

PRACTICAL ASTRONOMY.

In these notes it is proposed to set forth in outline the most useful methods for determining positions and directions on the surface of the earth. It is assumed that the observer is provided with an engineer's transit, or a nautical sextant, so that the methods described are only such as are adapted to the use of those instruments. More precise methods, necessitating instruments of the highest class, are therefore entirely omitted, or but briefly referred to.

1. Spherical Co-ordinates. Solution of the Astronomical Triangle.

Determination of the position of a point.

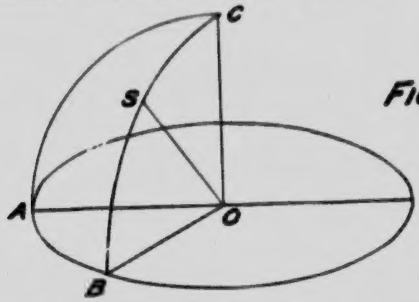


Fig. 1

In Fig. 1 *CAO* and *ABO* are fixed planes of reference; *O* is the point of observation The direction of the line *OS* is determined when the angles *AOB* and *BOS* are known; also when the spherical angle *ACS* and the arc *CS* are known.

Planes of reference—

The planes of reference used in astronomy are those of the equator, the ecliptic, the meridian, and the horizon.

The plane of the equator is that of the earth's equator. As the direction of the earth's axis is nearly fixed in space, being subject only to slow changes of direction due to precession and nutation, therefore the plane of the equator is nearly a fixed plane.

The plane of the ecliptic is the plane of the earth's orbit.

The plane of the observer's meridian is a plane determined by the earth's axis and the point of observation.

The plane of the horizon is a tangent plane to the earth's surface—*i.e.*, to the surface of standing water—at the point of observation. It is therefore perpendicular to the observer's plumb line.

The celestial sphere—

This is an imaginary sphere of infinite extent, whose centre is coincident with the centre of the earth. Upon its surface the heavenly bodies may be assumed to be, as they apparently are, set like brilliants.

The reference planes above defined are assumed to be produced to intersect this sphere in great circles. The plane of the horizon, as above defined, may be assumed to intersect the sphere in the same circle as that determined by a parallel plane through the earth's centre, owing to the infinite extent of the celestial sphere.

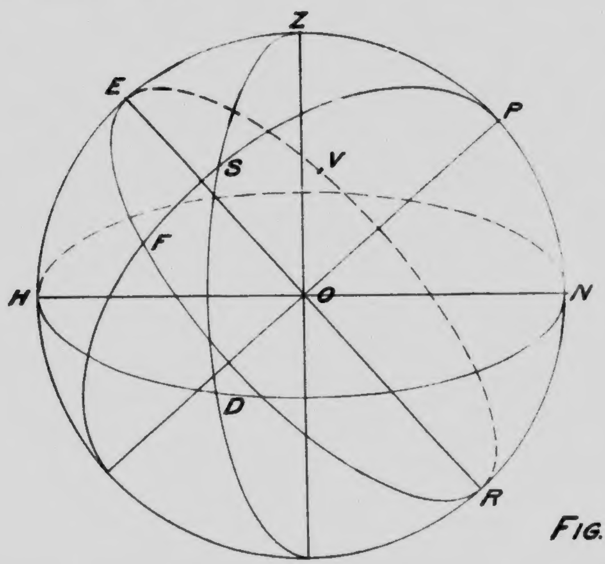


FIG. 2

Fig. 2 shews a projection of the celestial sphere on the plane of the meridian, the reference circles being represented. Thus

PZHR is the meridian,
EFR the equator, or equinoctial,
HDN the horizon.

The ecliptic is not shewn, but V is a point in which it intersects the equator.

If S now be the position of a star (by that term denoting any heavenly body), and secondaries ZSD and PSF to the horizon and equator respectively be drawn through it, these arcs, with the meridian PZ, form a spherical triangle PZS, which, from its frequent use in the solution of astronomical problems, is termed the ***astronomical triangle.***

Definitions—

The circle ZSD is a ***vertical circle***; PSF a ***declination*** or ***hour circle.*** P is the ***celestial pole***; Z the ***zenith.*** SD is the ***altitude*** of S; ZS its ***zenith distance***; SF its ***declination***; PS its ***polar distance***; the angle PZS its ***azimuth***; and ZPS its ***hour angle.*** PSZ is generally called the ***parallactic angle.***

As the observer's latitude is the angle between the direction of the plumb line at the place of observation and the plane of the equator, it follows that the latitude is the angle ZOE or the arc ZE. This is also equal to the arc PN.

The following notation will be used:

h denotes the altitude SD of S.
ζ denotes the ze ith dista ce ZS.
δ de otes the declination SF.
p denotes the polar distance PS.
τ denotes the hour angle ZPS.
A denotes the azimuth PZS.
C denotes the parallactic angle.
ϕ denotes the observer's latitude EZ or PN.
α denotes the right ascension VEF.

Systems of Spherical Co-ordinates.

1st system—Altitude and azimuth.

The arcs SD and DN serve to determine the position of S with reference to the horizon and the meridian.

A small circle parallel to the horizon is termed an almucantar.

A vertical circle is a great circle perpendicular to the horizon.

The prime vertical is that vertical circle which passes through the east and west points of the horizon.

2nd system—Declination and hour angle.

The arcs SF and FE determine the position of S with reference to the equator and the meridian.

A parallel of declination is a small circle parallel to the equator.

3rd system—Declination and right ascension.

The planes of the equator and the ecliptic intersect in a right line called the line of the equinoxes. This line intersects the sphere in the vernal and autumnal equinoxes. The vernal equinox is the point through which the sun passes in going from the south to the north side of the equator; it is shewn at V, Fig. 2.

The equinoctial colure is the declination circle passing through the equinoxes. The solstitial colure is the declination circle passing through the solstices—the points of greatest north and south declination on the ecliptic. It is therefore at right angles to the equinoctial colure.

The co-ordinates in this system are the arcs SF and FV.

4th system—Celestial latitude and longitude.

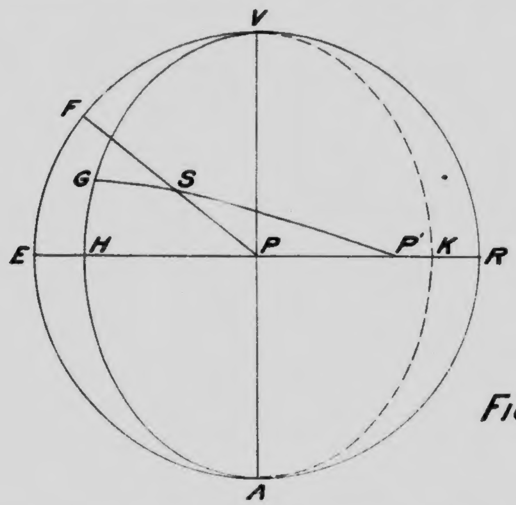


FIG. 3

In Fig. 3 $VEAR$ is the equator, $VHAK$ the ecliptic, VA the line of the equinoxes, VPA the equinoctial colure, and EPR the solstitial colure.

The co-ordinates in this system are SG, the latitude of S, and GV the longitude. These are denoted by β and λ respectively.

In the first system the co-ordinates change continually and irregularly on account of the diurnal rotation of the earth. In the second system the declination is unchanged by that rotation, and the hour angle changes uniformly with the time. In the third and fourth systems the co-ordinates are unchanged by the diurnal rotation.

The third system of co-ordinates is for this reason used in the construction of ephemerides.

Although unchanged by the diurnal rotation, the co-ordinates of the third and fourth systems are changing continually though slowly on account of precession and nutation.

Solution of the Astronomical Triangle.

(1) Given the altitude and azimuth of a star, and the latitude of the place, to find the star's declination and hour angle.

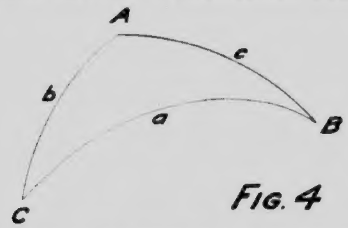


FIG. 4

If we denote the angular points of the astronomical triangle $Z\,P$ and S by $A\,B$ and C, respectively, then in Fig. 4 we have given

$$A = A,\ b = 90° - h,\ c = 90° - \phi;$$

and it is required to find

$$a = 90° - \delta, \text{ and } B = \tau.$$

These are given by the first of (1) and (5), *Sph. Trig.*, p. 69 which become

$$\sin\delta = \sin h \sin\phi + \cos h \cos\phi \cos A.$$

$$\sin A \cot\tau = \cos\phi \tan h - \sin\phi \cos A.$$

The first of these may be written

$$\sin\delta = \sin h\,(\sin\phi + \cot h \cos\phi \cos A)$$

Then introducing the auxiliary θ such that

$$\tan\theta = \cot h \cos A \qquad (1)$$

it becomes

$$\sin\delta = \sin h\,(\sin\phi + \cos\phi \tan\theta)$$

$$= \frac{\sin h \sin(\phi+\theta)}{\cos\theta} \qquad (2)$$

The second equation may be written

$$\tan\tau = \frac{\sin A}{\cos\phi \tan h - \sin\phi \cos A}$$

$$= \frac{\sin A}{\tan h(\cos\phi - \sin\phi \cot h \cos A)}$$

$$= \frac{\sin A}{\tan h(\cos\phi - \sin\phi \tan\theta)}$$

$$= \frac{\sin A \cos\theta}{\tan h \cos(\phi+\theta)}$$

Eliminating tan h by (1) this becomes

$$\tan \tau = \frac{\tan A \sin \theta}{\cos(\phi+\theta)} \tag{3}$$

Equations (1), (2) and (3) give the solution.

(2) Given the declination and hour angle of a star, and the latitude of the place, to find the altitude and azimuth of the star.

In the spherical triangle, Fig. 4, we have given

$$a = 90° - \delta,\ c = 90° - \phi, \text{ and } B = \tau$$

and $$b = 90° - h \text{ and } A$$

are required. These are given by the second equations of (1) and (5), *Sph. Trig.*, which become

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos \tau$$
$$\sin \tau \cot A = \cos \phi \tan \delta - \sin \phi \cos \tau$$

These may be written

$$\sin h = \sin \delta(\sin \phi + \cos \phi \cot \delta \cos \tau)$$
$$\tan A = \frac{\sin \tau}{\tan \delta(\cos \phi - \sin \phi \cot \delta \cos \tau)}$$

Then substituting $\cot \theta_1 = \cot \delta \cos \tau$ (4)

they become

$$\sin h = \frac{\sin \delta \cos(\theta_1 - \phi)}{\sin \theta_1} \tag{5}$$
$$\tan A = \frac{\sin \tau \sin \theta_1}{\tan \delta \sin(\theta_1 - \phi)}$$

Then eliminating tan δ from this last by (4) it becomes

$$\tan A = \frac{\tan \tau \cos \theta_1}{\sin(\theta_1 - \phi)} \tag{6}$$

θ_1 being given by the equation

$$\tan \theta_1 = \frac{\tan \delta}{\cos \tau} \tag{7}$$

These two problems serve for the transformation from the first system of co-ordinates to the second; and conversely.

(3) Given the altitude and declination of a star, and the latitude of the place, to find the azimuth and hour angle.

In this case we have given

$$a = 90° - \delta,\ b = 90° - h, \text{ and } c = 90° - \phi$$

and are required to find

$$A = A, \text{ and } B = \tau$$

These are given by the first and second of either set of equations (6), (7) or (8), *Sph. Trig.* In these equations we have

$$s \quad = \tfrac{1}{2}(a+b+c) \quad = 90° - \tfrac{1}{2}(\phi+\delta-\zeta)$$
$$s-a = \tfrac{1}{2}(-a+b+c) = \qquad \tfrac{1}{2}(\zeta+\delta-\phi)$$

$$s-b = \tfrac{1}{2}(a-b+c) = 90° - \tfrac{1}{2}(\zeta+\phi+\delta)$$
$$s-c = \tfrac{1}{2}(a+b-c) = \tfrac{1}{2}(\zeta+\phi-\delta)$$

so that on substituting $s' = \frac{1}{2}(\zeta+\phi+\delta)$ they become

$$\sin^2 \tfrac{1}{2}A = \frac{\cos s' \sin(s'-\delta)}{\cos\phi \sin\zeta} \tag{8}$$

$$\cos^2 \tfrac{1}{2}A = \frac{\cos(s'-\zeta)\sin(s'-\phi)}{\cos\phi \sin\zeta} \tag{9}$$

$$\tan^2 \tfrac{1}{2}A = \frac{\cos s' \sin(s'-\delta)}{\cos(s'-\zeta)\sin(s'-\phi)} \tag{10}$$

$$\sin^2 \tfrac{1}{2}\tau = \frac{\sin(s'-\phi)\sin(s_1-\delta)}{\cos\phi \cos\delta} \tag{11}$$

$$\cos^2 \tfrac{1}{2}\tau = \frac{\cos(s'-\zeta)\cos s'}{\cos\phi \cos\delta} \tag{12}$$

$$\tan^2 \tfrac{1}{2}\tau = \frac{\sin(s'-\phi)\sin(s'-\delta)}{\cos s' \cos(s'-\zeta)} \tag{13}$$

(4) Given the altitude, declination, and hour angle of a star, to find its azimuth, and the latitude of the place.

The data here are

$$a = 90° - \delta,\ b = 90° - h, \text{ and } B = \tau;$$

and the required quantities

$$A = A, \text{ and } c = 90° - \phi$$

These may be found by (3) and the second of (1), *Sph. Trig.*, which become

$$\sin A = \frac{\sin\tau \cos\delta}{\cos h} \tag{14}$$

$$\sin h = \sin\delta \sin\phi + \cos\delta \cos\phi \cos\tau$$

This last becomes (see eq. 5)

$$\sin h = \frac{\sin\delta \cos(\theta_1 - \phi)}{\sin\theta_1}$$

Then transposing, we have

$$\cos(\theta_1-\phi) = \frac{\sin h \sin\theta_1}{\sin\delta} \tag{15}$$

θ_1 being given by the eq.

$$\tan\theta_1 = \frac{\tan\delta}{\cos\tau}$$

There may be two solutions of this problem; but the ambiguity may be removed by first determining ϕ and then A by either of the equations (8), (9) or (10).

(5) Given the declination and azimuth of a star, and the latitude of the place, to find the hour angle and altitude.

Thus we have

$$a = 90° - \delta,\ A = A, \text{ and } c = 90° - \phi;$$

and are required to find

$$B=\tau, \text{ and } b=90°-h.$$

The first of these is given by the second of (5), *Sph. Trig.*, which becomes

$$\sin \tau \cot A=\cos \phi \tan \delta-\sin \phi \cos \tau$$

or $$\sin \tau \cot A+\sin \phi \cos \tau=\cos \phi \tan \delta$$

which may be thus transformed:

$$\cot A(\sin \tau+\tan A \sin \phi \cos \tau)=\cos \phi \tan \delta$$

or, substituting $\tan \theta_2=\tan A \sin \phi$ (16)

this becomes

$$\frac{\cot A \sin(\tau+\theta_2)}{\cos \theta_2}=\cos \phi \tan \delta$$

or, transposing

$$\sin(\tau+\theta_2)=\cos \phi \tan \delta \cos \theta_2 \tan A$$

Then eliminating $\tan A$ by (16) we have

$$\sin(\tau+\theta_2)=\cot \phi \tan \delta \sin \theta_2 \tag{17}$$

Equations (16) and (17) determine τ.

We may now find h by applying one of equations (3), *Sph. Trig.*, to the astronomical triangle, which gives

$$\cos h=\frac{\sin \tau \cos \delta}{\sin A} \tag{18}$$

We may also find h directly from the data by means of the first of (1), *Sph. Trig.*, which gives

$$\sin \delta=\sin h \sin \phi+\cos h \cos \phi \cos A;$$

which may be written

$$\sin \delta=\sin \phi(\sin h+\cos h \cot \phi \cos A);$$

in which substituting

$$\cot \theta_3=\cot \phi \cos A$$

we have

$$\sin \delta= \sin \phi(\sin h+\cos h \cot \theta_3) = \frac{\sin \phi \cos(h-\theta_3)}{\sin \theta_3}$$

$$\therefore \quad \cos(h-\theta_3)=\frac{\sin \delta \sin \theta_3}{\sin \phi} \tag{19}$$

Also $$\tan \theta_3=\frac{\tan \phi}{\cos A} \tag{20}$$

(6) To find the altitude, hour angle, and azimuth of a circumpolar star when at elongation, or maximum azimuth.

It is assumed that the latitude of the place is known. When a star is at elongation the angle C, Fig. 4, is a right angle, and the solution is given by equations (26), (28) and (27), *Sph. Trig.*, which become

$$\sin h=\frac{\sin \phi}{\sin \delta},\ \cos \tau=\frac{\tan \phi}{\tan \delta},\ \sin A=\frac{\cos \delta}{\cos \phi}. \quad (21), (22), (23)$$

(7) To find the altitude and hour angle of a star when on the prime vertical.

Here the azimuth A is equal to 90°, and it is assumed that ϕ and δ are given. Then applying equations (26) and (28), *Sph. Trig.*, we find

$$\cos \tau = \frac{\tan \delta}{\tan \phi} \qquad \sin h = \frac{\sin \delta}{\sin \phi} \tag{24), (25}$$

(8) Given the right ascension and declination of a star, and the obliquity of the ecliptic, to find the latitude and longitude of the star.

In the triangle $PP'S$, Fig. 3,

$$PS = 90° - \delta \qquad P'S = 90° - \beta$$
$$SPP' = 90° + \alpha \qquad SP'P = 90° - \lambda$$
$$PP' = \epsilon$$

and we have by equations (1), (4) and (3), *Sph. Trig.*,

$$\left.\begin{aligned} \sin \beta &= \sin \delta \cos \epsilon - \cos \delta \sin \epsilon \sin \alpha \\ \cos \beta \sin \lambda &= \sin \delta \sin \epsilon + \cos \delta \cos \epsilon \sin \alpha \\ \cos \beta \cos \lambda &= \cos \delta \cos \alpha \end{aligned}\right\} \tag{26}$$

Then substituting

$$\left.\begin{aligned} m \sin M &= \sin \delta \\ m \cos M &= \cos \delta \sin \alpha \end{aligned}\right\} \tag{27}$$

they become

$$\sin \beta = m \sin (M - \epsilon) \tag{28}$$
$$\cos \beta \sin \lambda = m \cos(M - \epsilon) \tag{29}$$

These may be written

$$\left.\begin{aligned} \tan M &= \frac{\tan \delta}{\sin \alpha} \\ \sin \beta &= \frac{\sin \delta \sin(M - \epsilon)}{\sin M} \\ \tan \lambda &= \frac{\tan \alpha \cos(M - \epsilon)}{\cos M} \end{aligned}\right\} \tag{30}$$

The quadrant in which M is situated is determined by equations (27), m being assumed always positive.

(9) Given the latitude and longitude of a star, and the obliquity of the ecliptic, to find the right ascension and declination of the star.

As in the last case we have

$$\left.\begin{aligned} \sin \delta &= \sin \beta \cos \epsilon + \cos \beta \sin \epsilon \sin \lambda \\ -\cos \delta \sin \alpha &= \sin \beta \sin \epsilon - \cos \beta \cos \epsilon \sin \lambda \\ \cos \delta \cos \alpha &= \cos \beta \cos \lambda; \end{aligned}\right\} \tag{31}$$

in which substituting

$$\left.\begin{aligned} n \sin N &= \sin \beta \\ n \cos N &= \cos \beta \sin \lambda \end{aligned}\right\} \tag{32}$$

they become

$$\sin\delta = n\sin(N+\epsilon) \tag{33}$$

$$\cos\delta\sin\alpha = n\cos(N+\epsilon) \tag{34}$$

$$\cos\delta\cos\alpha = \cos\beta\cos\lambda \tag{35}$$

From these we derive

$$\left.\begin{aligned}\tan N &= \frac{\tan\beta}{\sin\lambda}\\ \sin\delta &= \frac{\sin\beta\sin(N+\epsilon)}{\sin N}\\ \tan\alpha &= \frac{\tan\lambda\cos(N+\epsilon)}{\cos N}\end{aligned}\right\} \tag{36}$$

2. Time.

The sidereal day.

The earth's motion of rotation, as far as can at present be ascertained, is uniform; though theoretical considerations point to a possible retardation of its velocity. If such retardation exists, its amount must be extremely minute, as up to the present time none has been detected. The time of apparent rotation of the starry sphere is therefore sensibly constant, and may consequently be adopted as a unit of time and be denoted the sidereal day. Owing to the proper motions of the fixed stars the practical sidereal day is the time of rotation of the vernal equinox.

Sidereal time.

The sidereal day is assumed to begin at the instant of upper meridian transit of the vernal equinox, which point will in future be denoted by and referred to as the point V; and the sidereal time at any instant is the hour angle of V at that instant. It is thus equal to the right ascension of any star which is on the meridian of the observer at that instant.

The solar day.

A unit of time dependent on the sun is necessary for the purposes of daily life.

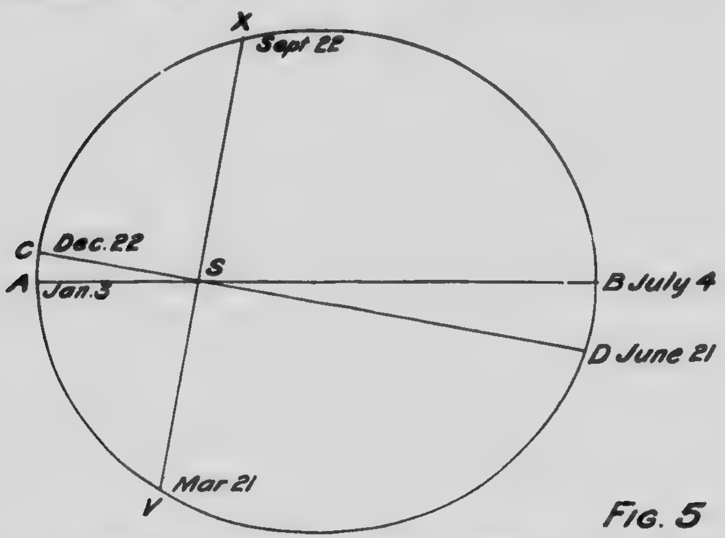


Fig. 5

On account of the earth's orbital motion about the sun the latter body has an apparent motion among the stars,

so that it returns to the meridian of a place nearly four minutes later on any given day than on the previous day, as shewn by a clock regulated to sidereal time.

This apparent motion of the sun, however, is not uniform. The earth moves in an ellipse, of which the sun occupies one of the foci, and its angular velocity about the sun varies inversely as the square of its radius vector; the angular velocity of the sun on the ecliptic therefore varies in the same manner. An inequality in the lengths of the solar days results from this; but a further irregularity is due to the obliquity of the ecliptic; for, even if the sun's motion on the ecliptic were uniform, its motion in right ascension would not be so.

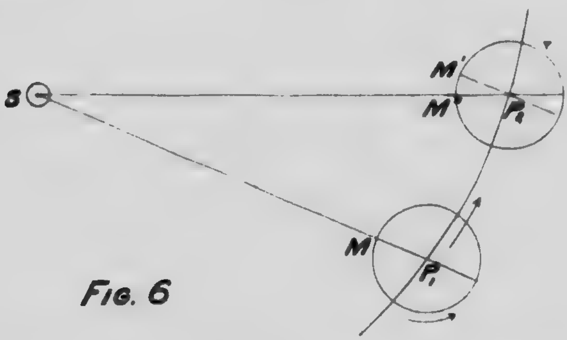


FIG. 6

This is illustrated in Fig. 6, which is a projection on a plane perpendicular to the earth's axis. P_1 and P_2 are two consecutive positions of the earth in which the sun is on a given meridian. The earth in the interval has performed a complete rotation on its axis plus the angle $M'P_2M''$, which equals P_1SP_2, which is the angle through which the projection of the radius vector has revolved during the interval. This angle varies from day to day, owing to the causes above mentioned, viz., the eccentricity of the earth's orbit and the obliquity of the ecliptic. The solar day, being equal in length to the time of an absolute rotation of the earth on its axis plus the variable angle P_1SP_2, is therefore variable in length. The angle P_1SP_2 is clearly the motion of the sun in right ascension in the solar day.

To obtain an invariable unit of time dependent upon the sun astronomers invented a fictitious sun, called the mean sun, and denoted by S_o in Fig. 7, which is assumed to move at a uniform rate on the equator and to return to the vernal equinox at the same instant as another fictitious sun S_1,

assumed to move at a uniform rate on the ecliptic. S_1 is also assumed to pass through perigee, and therefore apogee, at the same instant as the true sun.

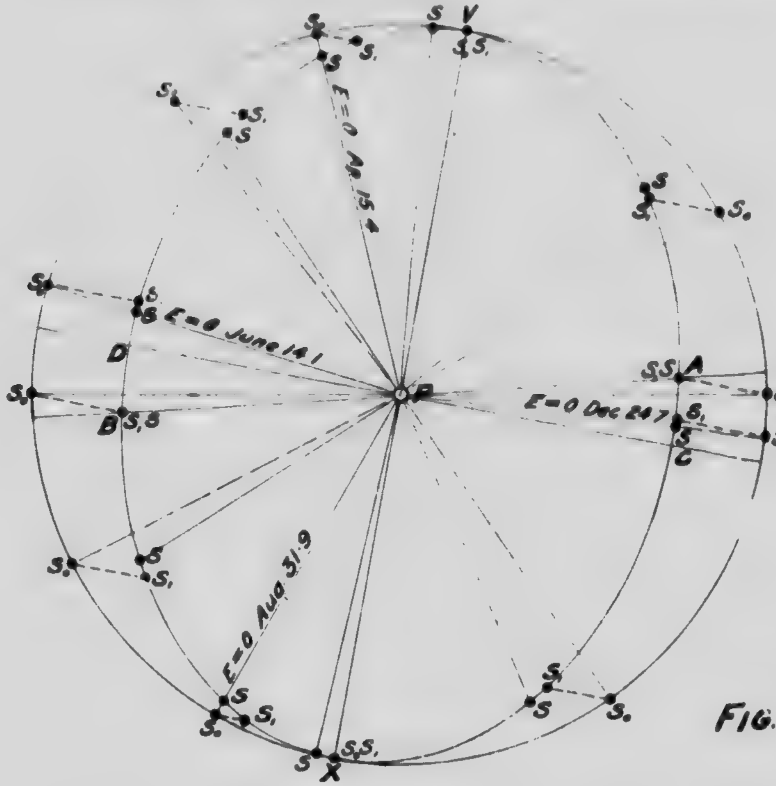


FIG. 7

The relative positions of the three suns at different times of the year are shewn in Fig. 7. There the points $VBXA$ shew the positions of the sun when the earth is at corresponding points in Fig. 5.

Solar time.

Apparent solar time at any instant is the hour angle of the true sun at that instant.

Mean solar time is the hour angle of the mean sun.

Apparent noon is the instant when the sun is on the meridian of a place. Mean noon is the instant when the mean sun is on the meridian.

The equation of time is the difference between apparent

2

and mean solar time; or, it is the difference of right ascension of the true and mean suns.

Tracing out the relative positions of the three suns in Fig. 7 shews that the equation of time changes its algebraic sign four times in the year, about April 15th, June 14th, Aug. 31st, and Dec. 24th. It therefore has four maximum values.

Civil and astronomical time.

The civil day begins at the instant of lower meridian transit of the mean sun, or at midnight; while the astronomical day of the same date begins at upper meridian transit 12 h. later.

Time at different meridians.

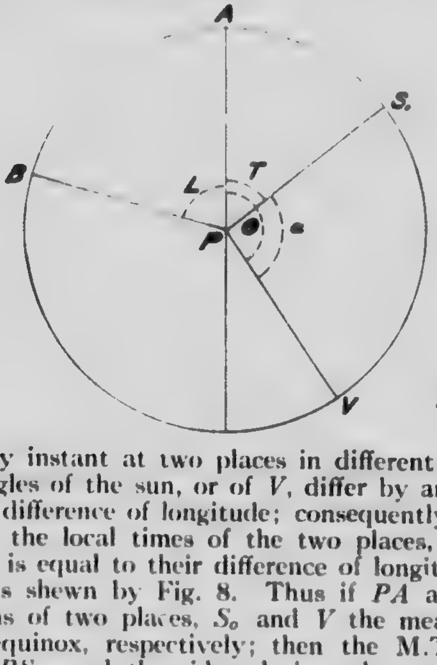


FIG. 8

At any instant at two places in different longitudes, the hour angles of the sun, or of V, differ by an amount equal to their difference of longitude; consequently the difference between the local times of the two places, either solar or sidereal, is equal to their difference of longitude.

This is shewn by Fig. 8. Thus if PA and PB are the meridians of two places, S_o and V the mean sun and the vernal equinox, respectively; then the M.T. at A is the angle APS_0, and the sidereal time the angle APV. The corresponding times at B exceed these by the angle APB (denoted by L).

Standard time.

For convenience, since 1883 the time used at any place in N. America, instead of being the local time of the place, is theoretically the time which differs by the nearest whole number of hours from Greenwich time. This is called stan-

dard time. Thus, the time which differs by 5^h from Gr. time is used at all points whose longitudes lie between 4^h 30^m and 5^h 30^m W. The following standard times are used in N. America:

Atlantic, differing by 4^h from Gr. time;
Eastern, differing by 5^h from Gr. time;
Central, differing by 6^h from Gr. time;
Mountain, differing by 7^h from Gr. time;
Pacific, differing by 8^h from Gr. time;
Yukon, differing by 9^h from Gr. time.

Relation between the lengths of the solar and sidereal units of time.

The tropical year is the interval of time between two consecutive passages of the mean sun through the mean vernal equinox

FIG. 9

In Fig. 9 let S and S' be the positions of the mean sun relatively to V at the instants of two consecutive transits over the meridian of some place. Then it is evident that the mean solar day is equal to the sidereal day plus the motion of the mean sun in right ascension in one mean solar day. (See also Fig. 6.) Therefore if

D = the length of the solar day, and
D' = the length of the sidereal day, and
n = the number of mean solar days in 1 tropical year;

then

$$\begin{aligned} 1 \text{ tropical year} &= nD \\ &= n\left(D' + \frac{1}{n} D'\right) \\ &= (n+1)D' \end{aligned}$$

But 1 tropical year = 365.24222 mean solar days

$\therefore$ 1 tropical year = 366.24222 sidereal days.

If then

M = any interval of time expressed in mean solar days,

S = the same interval expressed in sidereal days;

$$\therefore \qquad \frac{M}{S} = \frac{365.24222}{366.24222} = 1 - k, \text{ assume;}$$

and

$$\frac{S}{M} = \frac{366.24222}{365.24222} = 1 + k'$$

in which

$$k = 0.00273043$$
$$k' = 0.00273791$$

Also

$$24^h \text{ M.S.T.} = 24^h\ 03^m\ 56^s.555 \text{ Sid. T.}$$
$$24 \text{ Sid. T.} = 23\ \ 56\ \ 04\ .091 \text{ M.S.T.}$$

The conversion of a given interval of M.S.T. into the corresponding interval of Sid. T., or conversely, is best effected by means of tables given in the *Nautical Almanac*.

To convert the mean time at a given meridian into the corresponding sidereal time.

Let T = the given local M.T.;

Θ = the corresponding Sid. T.;

L = the longitude of the place;

V_o = the Gr. Sid. T. at the previous Gr. mean noon.

Then

$$\Theta = (T + L)\ (1 + k') + V_o - L \qquad (37)$$

V_o is taken from the ephemeris. Instead of using the factor k' the reduction of $T+L$ to the equivalent sidereal interval is made by means of tables given in the *N. A.*

To convert the sidereal time at a given meridian into the corresponding mean time.

Using the same notation, and in addition denoting by M the mean time at Gr. of the previous Gr. sidereal noon, we have

$$T = (\Theta + L)\ (1 - k) + M - L \qquad (38)$$

Here again the tables of the N. A. are used instead of the factor k.

The value of M, to be taken from the N. A., is to be that for the date of the transit of V immediately preceding the given time. Thus if

$$(\Theta + L)\ (1 - k) + M > 24^h$$

then the value of M must be taken for the previous date.

To convert the apparent solar time at a given meridian into the corresponding sidereal time.

This may be done by first reducing to M.T. by applying the equation of time—to be taken from the N. A.—and then reducing to sidereal time by the method given above.

A more convenient method, however, is to interpolate from the N. A. the value of the sun's right ascension at the Gr. instant corresponding to the given time. Then if in Fig. 8 S_0 represents the true sun it is clear that if t = the hour angle of the sun, or the apparent time, then

$$\Theta = t + \alpha \qquad (39)$$

To determine the hour angle of a heavenly body at a given time.

If in (39) it is assumed that the hour angle t may have any value up to 24^h, then that equation is general and applies to every case and any heavenly body. It may be necessary in some cases to deduct 24^h from $t + \alpha$. Transposing we have

$$t = \Theta - \alpha \qquad (40)$$

Here it may be necessary in some cases to increase Θ by 24^h to render subtraction possible.

The hour angle denoted by τ, found by solving the astronomical triangle—the parts of that triangle being limited to values less than 180°—, being given, we have

$$t = \tau \text{ if west}$$
$$t = 24^h - \tau \text{ if east.}$$

The hour angle of the sun may be found by equation (40) if the sidereal time is given. If the mean time is given, it may be reduced to apparent time by applying the equation of time, thus finding the required hour angle.

Reduction of time to arc; and conversely.

These reductions may be made by means of the following numerical relations:

$$1^h = 15^\circ \qquad 1^\circ = 4^m$$
$$1^m = 15' \qquad 1' = 4^s$$
$$1^s = 15''$$

3. Determination of Time by Observation.

Correction and rate of a chronometer.

As the term implies the correction of a chronometer is the amount that must be *added* to the chronometer time to give the true time.

The rate of a chronometer is the amount it *loses* in a unit of time.

Thus, if

T_1 and T_2 = the true times at given instants;
T'_1 and T'_2 = the chronometer time at those instants;
ΔT_1 and ΔT_2 = the chronometer corrections;
δT = the chronometer rate.

Then

$$\left.\begin{aligned}\Delta T_1 &= T_1 - T_1' \\ \Delta T_2 &= T_2 - T_2'\end{aligned}\right\} \tag{41}$$

$$\delta T = \frac{\Delta T_2 - \Delta T_1}{T_2' - T_1'} \tag{42}$$

These equations give the corrections and rate with their proper algebraic signs. The rate is thus given in terms of the chronometer interval.

1st method—By transits.

(*a*) *Meriuian transits.*

A transit instrument having been adjusted in the meridian, the time of transit of any heavenly body across the ' . wire may be observed by a chronometer whose correcti is to be found.

If the chronometer is regulated to sidereal time the true sidereal time of transit is at once given by the right ascension of the body, whence the chronometer correction at once follows by (41). If regulated to mean time, the sidereal time of transit of the body must be reduced to mean time.

If the sun is observed, the time of transit of each limb should be noted and the mean taken; thus finding the time of transit of the centre. If only one limb can be observedf then the observed time must be corrected by the "time o, semi-diameter passing the meridian", which may be taken from the N. A., or computed by the equation

$$\delta t = \frac{S}{15} \sec \delta \tag{43}$$

in which S is the angular semi-diameter of the sun.

If the correction of a M.T. chronometer is to be found by a transit of the sun, the true M.T. of transit may at once be found by applying the equation of time to the apparent time of transit 0^h.

(*b*) *Transits across any vert. circle of known azimuth.*

In th' case the latitude of the place and the declination of the heavenly body must be known; then the hour angle may be computed by means of (16) and (17), which may be written

$$\tan \theta = \tan A \sin \phi$$
$$\sin(\tau + \theta) = \cot \phi \tan \delta \sin \theta$$

The sidereal time then follows by (39), or the M.T. by applying the equation of time as already shewn.

The rate of a chronometer may be found by observing two consecutive transits of a star across the same vert. circle. The true interval between the transits is

$$24^h \text{ Sid. T. or } 23^h\ 56^m\ 04^s.09 \text{ M.T.}$$

(*c*) *Transits across the vertical circle of Polaris.*

This method will be described under Azimuth.

2nd method—By a single altitude.

The method of observing an altitude of a heavenly body is described below, p. 65 et seq.

Corrections to be applied to an Observed Altitude.

(*a*) *Refraction.*

The ray of light that reaches an observer from a star, in traversing the earth's atmosphere is continually bent downwards from a rectilinear path by the increasing refractive power of the air with density as the surface of the earth is approached. In consequence, the apparent direction of a

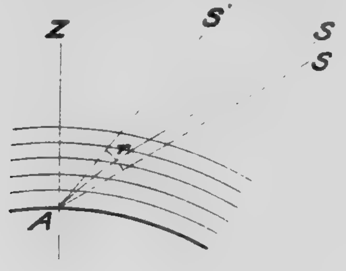


Fig. 10

star is that of a tangent to the curved path of the ray at the point where it reaches the observer. This is illustrated in Fig. 10.

An observed altitude must then be diminished by an amount equal to the angle between the final direction of the ray and the straight line drawn to the star, as appears in the figure. The magnitude of r decreases as the altitude increases, and its value is best found from tables. These contain corrections depending upon the readings of the barometer and thermometer. An approximate value of r may be found by the equation

$$r = 57''.7 \tan \zeta$$

or a closer approximation by the formula

$$r = \frac{983b}{460 + t} \tan \zeta$$

in which

b = the barometer reading in inches; and

t = the temperature of the air in degrees F.

(See *Field Astronomy for Engineers*, by Prof. G. C. Comstock).

(*b*) *Semi-diameter.*

In observing the sun or moon the altitude of its upper or lower limb is observed. To find the altitude of its centre a correction for semi-diameter must be applied. This may be found in the N. A.

(*c*) *Parallax.*

As the centre of the celestial sphere is coincident with that of the earth, if the directions of a heavenly body from that point and from a point on the earth's surface differ sensibly,

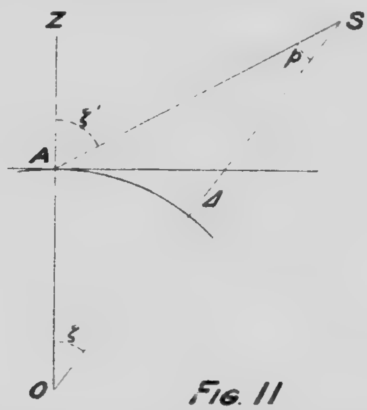


FIG. 11

then a correction must be applied to any observed co-ordinate to reduce it to the centre of the earth. This is only necessary with members of the solar system.

In Fig. 11 S is the centre of the heavenly body observed, O the centre of the earth, A the point of observation. The triangle ASO gives

$$\sin p = \sin \zeta' \frac{a}{\Delta}$$

p being the parallax in altitude. If $\zeta' = 90°$ the resulting value of p is the horizontal parallax. Denoting it by π we have

$$\sin \pi = \frac{a}{\Delta}$$

$$\therefore \qquad \sin p = \sin \pi \sin \zeta'; \tag{44}$$

or very nearly

$$p = \pi \sin \zeta' = \pi \cos h' \tag{45}$$

This gives the correction for parallax with sufficient accuracy for any body except the moon.

(*d*) *Dip of the horizon.*

At sea the altitude of a heavenly body is measured with a sextant from the sea horizon, the observer standing on the deck of a ship. A correction must therefore be applied to the observed angle on account of the dip of the visible below the true horizon.

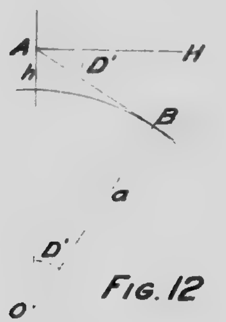


FIG. 12

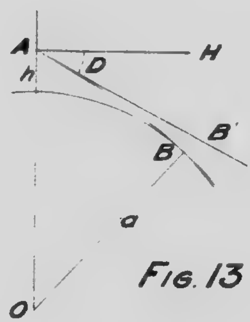


FIG. 13

In Fig. 12 we have from Pl. Geom.

$$\tan D' = \frac{AB}{a} = \frac{\sqrt{(2a+h)h}}{a} = \frac{\sqrt{2ah}}{a} \text{ nearly}$$

or

$$D' = \sqrt{\frac{2h}{a}}$$

This gives the dip uncorrected for refraction; but, as shewn in Fig. 13, the ray of light which reaches the observer from the horizon follows a curved path, so that the apparent dip

D is less than D'. The mean value of the ratio of D to D' is .9216 : 1, so that

$$D = .9216\sqrt{\frac{2h}{a}}$$

or in seconds of arc

$$D = \frac{.9216}{\sin 1''}\sqrt{\frac{2h}{a}}$$

Substituting a mean value of a in feet, this becomes

$$D = 58''.82\sqrt{h} \tag{46}$$

h being in feet.

The rule known to navigators: "Take the square root of the height of the eye above sea level in feet and call the result minutes", is thus very approximately correct.

Having applied the necessary corrections to the observed altitude, the reduction may be made by either of the equations (11), (12) or (13). If a number of observations are to be reduced an equation derived as follows is more convenient: Taking the equation

$$\cos\zeta = \sin\delta\sin\phi + \cos\delta\cos\phi\cos\tau$$

it may be written

$$\begin{aligned} 1 - \text{versin}\,\zeta &= \sin\delta\sin\phi + \cos\delta\cos\phi(1 - \text{versin}\,\tau) \\ &= \cos(\phi-\delta) - \cos\phi\cos\delta\,\text{versin}\,\tau \\ &= 1 - \text{versin}(\phi-\delta) - \cos\phi\cos\delta\,\text{versin}\,\tau \end{aligned}$$

$$\therefore \quad \text{versin}\,\tau = \frac{\text{versin}\,\zeta - \text{versin}\,(\phi-\delta)}{\cos\phi\cos\delta} \tag{47}$$

This requires the use of tables of natural and logarithmic versins. In the absence of such a table the following form of the equation may be used

$$\sin^2\tfrac{1}{2}\tau = \frac{\cos(\phi-\delta) - \cos\zeta}{2\cos\phi\cos\delta} \tag{48}$$

Example.—The following observations were taken with a sextant and artificial horizon on Aug. 1, 1892, at a place in latitude 52° 31′ 04″, and approximate longitude $7^h\ 50^m$ W.; to find the watch correction.

$2-$*alt.* $\overline{\odot}$	*Watch*
52° 11′ 30″	$7^h\ 21^m\ 29^s$ A.M.
52 38 10	22 54
53 05 30	24 27
53 24 30	25 28
54 16 00	28 18
54 44 10	29 52
55 03 10	30 54

Index error = +20″.

First find the approximate Gr. M.T., thus:

Mean of extreme times	$= 7^h\ 26^m\ 11^s$
Ast. time, July 31	$= 19\ \ 26\ \ 11$
Long	$= 7\ \ 50$
Gr. M.T., Aug. 1	$= 3\ \ 16\ \ 11$

For this time we take from the N. A.

$$\delta = +17^\circ\ 48'\ 56''$$
$$S = 15\ \ 48$$
$$E = 6^m\ 03^s.6$$

Reduction of first observation:

Obs'd. 2 − alt.	$= 52^\circ\ 11'\ 30''$
Index error	$+20$
	$2)52\ \ 11\ \ 50$
h'	$= 26\ \ 05\ \ 55$
r	$= 1\ \ 52$
	$26\ \ 04\ \ 03$
S	$= 15\ \ 48$
	$25\ \ 48\ \ 15$
p	$= 8$
h	$= 25\ \ 48\ \ 23$
ζ	$= 64\ \ 11\ \ 37$

Eq. (13)	s'	$= 67^\circ\ 15'\ 48''.5$
	$s'-\phi$	$= 14\ \ 44\ \ 44\ .5$
	$s'-\delta$	$= 49\ \ 26\ \ 52\ .5$
	$s'-\zeta$	$= 3\ \ 04\ \ 11\ .5$
	$\log \sin (s'-\phi)$	$= 9.405738$
	$\log \sin (s'-\delta)$	$= 9.880708$
	$\log \cos s'$	$= 9.587143$
	$\log \cos (s'-\zeta)$	$= 9.999377$
		9.286446
		9.586520
	$\log \tan^2 \frac{1}{2}\tau$	$= 9.699926$
	$\log \tan \frac{1}{2}\tau$	$= 9.849963$
	$\frac{1}{2}\tau$	$= 35^\circ\ 17'\ 38''.9$
	τ	$= 70\ \ 35\ \ 17\ .8$
		$= 4^h\ 42^m\ 21^s.2$

∴ App. Time	=	7	17	38	.8
E	=		6	03	.6
Mean Time	=	7	23	42	.4
Watch	=	7	21	29	
ΔT	=		2	13	.4

Having reduced the remaining observations the complete results are as follows:

ΔT
$+2^m\ 13^s .4$
16 .0
13 .2
15 .0
15 .3
14 .3
15 .2
Mean = +2 14 .6

Another example will be found on p. 43.

To find the effect of errors in the data on the time computed from an observed altitude.

Taking the equation (see Fig. 4):

$$\cos b - \cos a \cos c - \sin a \sin c \cos B = 0$$

and differentiating by means of the expression

$$-\frac{df}{dB}dB = \frac{df}{da}da + \frac{df}{db}db + \frac{df}{dc}dc$$

we find

$$\begin{aligned} -\sin a \sin c \sin B dB = &\\ &(\sin a \cos c - \cos a \sin c \cos B)da \\ &-\sin b db + (\cos a \sin c + \sin a \cos c \cos B)dc \\ &= \sin b \cos C da - \sin b db + \sin b \cos A dc \end{aligned}$$

by applying equations (4), *Sph. Trig.* Then substituting in the left-hand number

$$\sin a \sin B = \sin b \sin A$$

we have

$$-dB = \frac{\cos C da}{\sin c \sin A} - \frac{db}{\sin c \sin A} + \frac{dc}{\sin c \tan A}$$

Then introducing the astronomical co-ordinates, and remembering that

$$da = -d\delta \quad db = -dh \quad dc = -d\phi$$

we have finally

$$d\tau = \frac{\cos C d\delta}{\cos\phi \sin A} - \frac{dh}{\cos\phi \sin A} + \frac{d\phi}{\cos\phi \tan A} \tag{49}$$

The errors being small may be regarded as differentials, so that (49) gives the effect of errors in δ, h, and ϕ upon the resulting hour angle τ. It shews moreover that the effect of those errors is least when A and C are both large, or when the star observed is near the prime vertical.

3rd method—By equal altitudes of a heavenly body.

Method of observation with a transit or sextant.

Equal altitudes of a heavenly body on opposite sides of the meridian correspond, generally speaking, to equal hour angles. This is the case of a fixed star, whose change of declination between the two positions may be neglected. The mean of the times of equal altitudes is then the time of meridian transit. The method is therefore an indirect one for observing the time of meridian transit.

In the case of the sun, however, allowance must be made for the change of declination in the interval between the two observations. An expression for the correction to be applied to the mean of the observed times is derived as follows:

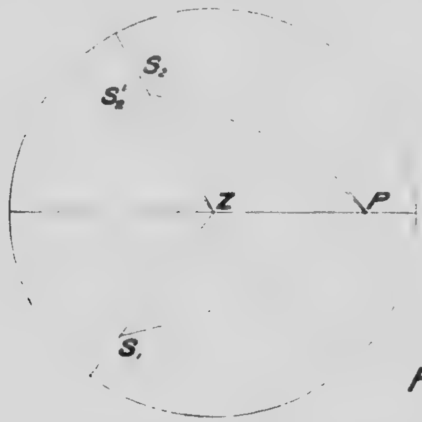


FIG. 14

Fig. 14 shews a projection of the celestial sphere on the plane of the horizon. S_1 and S_2 are the two positions of the sun's centre at the instants of the two observations; S'_2 the position it would have occupied if there had been no change of declination. The two triangles PZS_1 and PZS'_2 are then equal in all respects. It is therefore required to find the change of hour angle resulting from a small change of declination. Taking the equation

$$\cos \zeta - \sin \delta \sin \phi - \cos \delta \cos \phi \cos \tau = 0$$

we find by differentiation

$$\frac{d\tau}{d\delta} = \frac{\cos\delta\sin\phi + \sin\delta\cos\phi\cos\tau}{\cos\delta\cos\phi\sin\tau}$$

$$= \frac{\tan\phi}{\sin\tau} + \frac{\tan\delta}{\tan\tau}$$

If we now write

$$d\tau = -2\Delta T_o \qquad d\delta = 2\Delta\delta$$

this becomes

$$-2\Delta T_o = \left(\frac{\tan\phi}{\sin\tau} + \frac{\tan\delta}{\tan\tau}\right).2\Delta\delta$$

or in seconds of time

$$\Delta T_o = -\frac{\Delta\delta}{15}\left(\frac{\tan\phi}{\sin\tau} + \frac{\tan\delta}{\tan\tau}\right) \tag{50}$$

This is the "equation of equal altitudes."

In this equation

ΔT_o = the correction to be applied to the mean of the observed times to find the time of meridian transit;

$\Delta\delta$ = half the change in the sun's declination in the interval between the observations, positive if the sun is moving north.

τ may be assumed equal to half the elapsed interval between the observations. Attention must be paid to the algebraic sign of δ; it is positive if north.

The advantages of this method are that the absolute altitudes need not be known; and small errors in ϕ and δ have no appreciable effect.

To find the time of rising or setting of a heavenly body.

Take the equation

$$\sin h = \sin\delta\sin\phi + \cos\delta\cos\phi\cos\tau;$$

which may be written

$$\cos\tau = \sin h\sec\phi\sec\delta - \tan\phi\tan\delta \tag{51}$$

In the case of the sun, when its upper limb is just visible in the horizon it is in reality below the horizon by the amount of the refraction, 34′ approximately; and its centre is below the limb by the amount of the semi-diameter, which may be taken as 16′; parallax may be neglected. Therefore $h = -50'$, and $\sin 50' = 0.0145$; $\therefore$ the above equation becomes

$$\cos\tau = -0.0145\sec\phi\sec\delta - \tan\phi\tan\delta \tag{52}$$

The time of rising of the moon's centre is usually computed. In this case the effect of parallax is important. Assuming its amount as 57′, the altitude of the moon's centre when it is

apparently in the horizon $= 57' - 34' = 23'$. Also $\sin 23' = 0.0067$; so that (51) becomes

$$\cos \tau = 0.0067 \sec \phi \sec \delta - \tan \phi \tan \delta \qquad (53)$$

Having computed the hour angle, the time readily follows.

Construction of sun dials.

The horizontal dial and the prime vertical dial only will be considered.

In any form of dial the edge of the gnomon which casts the shadow must be parallel to the earth's axis, as the position of the shadow cast upon any plane is then independent of the sun's declination

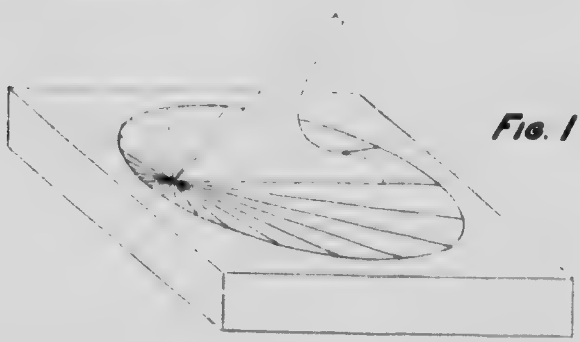


FIG. 15

Fig. 15 shews the construction of the horizontal dial. The edge of the gnomon if produced will intersect the celestial

FIG. 16

sphere in the pole P, Fig. 16. PON is the meridian plane, NOL a horizontal plane, and POL a plane through the sun's centre. LON (denoted by a) is the angle which an hour line, corresponding to a given hour angle τ, makes with the noon line. The triangle PLN then gives

$$\tan a = \sin \phi \tan \tau \qquad (54)$$

The construction for a prime vertical dial is shewn in Fig. 17. OPZ' is the meridian plane; OMZ' that of the

Fig. 17

prime vertical; and $OP'M$ a plane through the sun's centre. β is the required angle corresponding to the hour angle τ. The triangle $P'MZ'$ gives

$$\tan \beta = \cos \phi \tan \tau \tag{55}$$

A sun dial gives apparent solar time.

4. Determination of Latitude by Observation.

As shewn on p. 3, the latitude of a place is equal to the altitude of the pole, or the declination of the zenith, *i.e.*, to either arc PN or EZ, Fig. 2.

1st method—By meridian altitudes or zenith distances.

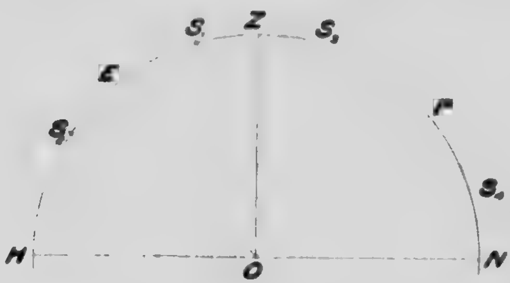


Fig 18

If the altitude or zenith distance of a heavenly body be observed when crossing the meridian, and the necessary corrections be applied, the latitude at once follows by one of the following equations, depending upon the position of the body. For the star

$$\left.\begin{array}{l} S_1 \ldots \phi = \zeta + \delta \\ S_2 \ldots \phi = \zeta + \delta \ (\delta \text{ being negative}) \\ S_3 \ldots \phi = \delta - \zeta = h - p \\ S_4 \ldots \phi = 180^\circ - \delta - \zeta = h + p \end{array}\right\} \qquad (56)$$

If S_3 and S_4 are the positions of the same star observed at both culminations, then by taking the mean

$$\phi = \frac{h + h'}{2} - \frac{p - p'}{2} \qquad (57)$$

the accented letters belonging to lower culmination.

If S_1 and S_3 are two stars observed at nearly equal zenith distances, we have by taking the mean of the first and third of (56)

$$\phi = \frac{\delta + \delta'}{2} + \frac{\zeta - \zeta'}{2} \qquad (58)$$

the accented letters belonging to the north star. This formula is the basis of Talcott's method of determining latitude, the observed quantity being the difference of zenith distance of the two stars, which are selected so that that difference is small enough to be measured by a filar micrometer placed in the focus of a telescope. Details of method outlined.

If the direction of the meridian is not known the maximum altitude of the heavenly body may be observed. If that body is the sun the maximum altitude differs slightly from the meridian altitude, owing to its rapidly changing declination. The resulting error is entirely negligible, especially if instruments of only moderate precision are used; its value is given by the expression

$$\left(\frac{\Delta\delta}{54000}\right)^2 \cdot \frac{\tan\phi - \tan\delta}{2\sin 1''} \quad \text{or } [5.54861]\,(\Delta\delta)^2\,(\tan\phi - \tan\delta)$$

in which $\Delta\delta$ is the hourly change in the declination expressed in seconds. The correction is always subtractive.

Example.—On July 10, 1914, the meridian altitude of the sun's upper limb was observed (Cir. I) to be:

68° 11′ 30″.

To find the index error of the transit used the following V.C.R's were taken on a terrestrial point:

Cir. L......		0° 34′ 30″
Cir. R.......		0 31
Diff.	=	3 30
I. E.	=	1 45
Obs'd alt.	=	68° 11′ 30″
I. E.	=	1 45
h'	=	68 09 45
r	=	23
		68 09 22
S	=	15 46
		67 53 36
p	=	3
h	=	67 53 39
ζ	=	22 06 21
δ	=	22 17 48
ϕ	=	44 24 09

2nd method—By an altitude observed out of the meridian, the time being known.

To the observed altitude the necessary corrections must be applied, and the hour angle derived from the observed time. The latitude then follows by means of (15)

$$\cos(\phi-\theta)=\frac{\sin h\sin\theta}{\sin\delta}$$

θ being found by the equation

$$\tan\theta=\frac{\tan\delta}{\cos\tau}$$

To find the effect of errors in the data we have by transposing (49)

$$d\phi=-\cos C\sec A d\delta+\sec A dh+\cos\phi\tan A d\tau \qquad (59)$$

This equation shews that the effect of errors in the data is least when A is small and C large, though the second condition is unimportant, as the error in the declination is always small in comparison with the other errors. These conditions are fulfilled, however, by observing a close circumpolar star near elongation.

Hence the method by means of the pole star.

As the altitude of this star never differs much from the latitude, the method consists in computing a correction to apply to the forms to give the latter. An expression for this correction is derived as follows:

Taking the equation

$$\sin h=\sin\phi\sin\delta+\cos\phi\cos\delta\cos\tau$$

and substituting in it

$$\phi=h+x$$
$$\delta=90^\circ-p$$

we have

$$\sin h=\sin(h+x)\cos p+\cos(h+x)\sin p\cos\tau$$

Then expanding the sin and cos of $h+x$, and again expanding the sin and cos of x and p and neglecting the powers of their circular measures above the second, we have

$$\begin{aligned}\sin h &= \left\{\sin h\left(1-\frac{x^2}{2}\right)+x\cos h\right\}\left(1-\frac{p^2}{2}\right)\\ &\quad+\left\{\cos h\left(1-\frac{x^2}{2}\right)-x\sin h\right\}p\cos\tau\\ &=\sin h-\frac{x^2}{2}\sin h+x\cos h-\frac{p^2}{2}\sin h+p\cos\tau\cos h\\ &\quad-px\cos\tau\sin h.\end{aligned}$$

Whence

$$x\cos h=-p\cos h\cos\tau+\tfrac{1}{2}(x^2+p^2+2px\cos\tau)\sin h$$

or

$$x=-p\cos\tau+\tfrac{1}{2}(x^2+p^2+2px\cos\tau)\tan h.$$

Assuming now as a first approximation

$$x = -p \cos \tau,$$

and substituting in the right-hand member, we have

$$x = -p \cos \tau + \tfrac{1}{2}(p^2 \cos^2 \tau + p^2 - 2p^2 \cos^2 \tau) \tan h$$
$$= -p \cos \tau + \tfrac{1}{2}p^2 \sin^2 \tau \tan \mathrm{h}$$

or in seconds of arc

$$x = -p \cos \tau + \tfrac{1}{2}p^2 \sin 1'' \sin^2 \tau \tan h$$

We have then finally

$$\phi = h - p \cos \tau + \tfrac{1}{2}p^2 \sin 1'' \sin^2 \tau \tan h \qquad (60)$$

The effect of the omission of the smaller terms in the above expansions can never amount to $0''.5$.

Example.—The following observations of Polaris were taken on June 14, 1904, with a small transit:

Cir.	*V. C. R.*	*Watch*
R.	45° 44′	$14^h\ 50^m\ 04^s$
L.	45 43	53 46
R.	45 45	57 10
L.	45 44	59 44

The watch was regulated to sid. time, and its correction was -20^s. The star's co-ordinates were:

$$\alpha = 1^h\ 24^m\ 26^s$$
$$\delta = 88°\ 47'\ 27''\ (\therefore p = 4353'').$$

The mean of the first and second observations being taken, and that of the third and fourth, the reduction is made as follows:

Eq. (60)	T'	$= 14^h\ 51^m\ 55^s$	$= 14^h\ 58^m\ 27^s$
	ΔT	$= -20$	$= -20$
	Θ	= 14 51 35	= 14 58 07
	α	= 1 24 26	= 1 24 26
	t	= 13 27 09	= 13 33 41
	τ	= 10 32 51	= 10 26 19
		= 158° 12′ 45″	= 156° 34′ 45″
	h'	= 45 43 30	45 44 30
	r	= 56	= 56
	h	= 45 42 34	= 45 43 34
	log p	= 3.638789	= 3.638789
	log cos τ	= 9.967813n	= 9.962659n
	log 1st term	= 3.606602n	= 3.601448n

log 0.5	=	$\bar{1}.698970$	=	$\bar{1}.698970$
log p^2	=	$\bar{7}.277578$	=	$\bar{7}.277578$
log sin 1″	=	$\bar{6}.685575$	=	$\bar{6}.685575$
log $\sin^2 \tau$	=	9.139134	=	9.198634
log tan h	=	10.010756	=	10.011009
log 2nd term	=	0.812013	=	0.871766
h	=	45° 42′ 34″	=	45° 43′ 34″
1st term	=	1 07 22	=	1 06 34
2nd term	=	6	=	7
ϕ	=	46 50 02		46 50 15

Mean = 46° 50′ 08″

Circum-meridian Altitudes.

If a number of altitudes of a star be observed in quick succession when near the meridian, each will differ by but a small amount from the meridian altitude. A correction may then be computed for each altitude which, when applied to it will give a value of the meridian altitude. The mean of these resulting values having been taken the latitude then follows by means of one of the equations (56).

To find an expression for this correction we return to the equation

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos \tau,$$

which is transformed as follows:

$$\begin{aligned} \sin h &= \sin \phi \sin \delta + \cos \phi \cos \delta (1 - 2 \sin^2 \tfrac{1}{2} \tau) \\ &= \cos(\phi - \delta) - \cos \phi \cos \delta \,.\, 2 \sin^2 \tfrac{1}{2} \tau \\ &= \cos \zeta_o - \cos \phi \cos \delta \,.\, 2 \sin^2 \tfrac{1}{2} \tau \\ &= \sin h_o - \cos \phi \cos \delta \,.\, 2 \sin^2 \tfrac{1}{2} \tau \end{aligned}$$

by (56), ζ_o being the meridian zenith distance and h_o the meridian altitude. If we now write

$$h = h_o - y$$

we have $\sin h = \sin(h_o - y) = \sin h_o - y \cos h_o$

by expanding and discarding powers of y above the first. Substituting in the above expression for $\sin h$, it becomes

$$\sin h_o - y \cos h_o = \sin h_o - \cos \phi \cos \delta \,.\, 2 \sin^2 \tfrac{1}{2} \tau$$

or

$$y = \frac{\cos \phi \cos \delta}{\cos h_o} \,.\, 2 \sin^2 \tfrac{1}{2} \tau$$

or in seconds of arc

$$y = \frac{\cos \phi \cos \delta}{\cos h_o} \cdot \frac{2 \sin^2 \tfrac{1}{2} \tau}{\sin 1''} \tag{61}$$

This gives the required correction.

If the squares of small quantities be retained in the above expansions the following term will be added to (61)

$$-\left(\frac{\cos\phi\ \cos\delta}{\cos h_o}\right)^2 \tan h_o\ \frac{2\sin^4\frac{1}{2}\tau}{\sin 1''}$$

The value of the term

$$\frac{2\sin^4\frac{1}{2}\tau}{\sin 1''}$$

amounts to $1''$ for $\tau = 18^m$, and to $7''.55$ when $\tau = 30^m$, so that for moderate hour angles, and when using small instruments, (61) may be considered practically exact.

Many collections of tables give the values in seconds of arc of the terms

$$m = \frac{2\sin^2\frac{1}{2}\tau}{\sin 1''} \text{ and } n = \frac{2\sin^4\frac{1}{2}\tau}{\sin 1''}$$

for given values of τ.

Example.—The following observations were taken with a sextant and artificial horizon on Sept. 2, 1893:

2−alt. ☉	*Watch*
89° 59′ 15″	$11^h\ 53^m\ 36^s$
90 00 15	56 37
90 00 45	59 28
89 59 15	12 03 57
89 58 30	05 46
89 57 30	07 11
89 55 15	09 13

Index error = 0; watch correction = -8^s.

An approximate value of the latitude is found by regarding the maximum observed altitude as the meridian altitude, as follows:

Max. 2−alt.	= 90° 00′ 45″
Eccentric error	+2 00
	90 02 45
Obs'd alt.	= 45 01 22
r	= 58
	45 00 24
S	= 15 54
	45 16 18
p	= 6
h_o	= 45 16 24

ζ_o	=	44	43	36
δ	=	7	37	54
ϕ (approx.)	=	52	21	30

The hour angles corresponding to the observed times may be found by first finding the watch time of culmination, thus

App. time of culm'n	=	12^h	00^m	00^s
E	=			-21
M.T.	=	11	59	39
ΔT	=			-08
Watch time of culm'n	=	11	59	47

From this follow the hour angles tabulated below. The corrected zenith distances are also found as above. We then proceed as follows:

$$\log \cos \phi = 9.785843$$
$$\log \cos \delta = 9.996136$$
$$\log \cos h_o = 9.847403$$
$$9.781979$$
$$9.934576$$
$$\log m = 1.87545$$
$$\log (h_o - h) = 1.81003$$
$$h_o - h = 64''.57$$

The remaining corrections are comput·d in a similar manner, and are tabulated below.

ζ	τ	$h_o - h$	ζ_o
44° 44′ 21″	6^m 11^s	1′ 05″	44° 43′ 16″
43 51	3 10	17	34
43 36	0 19	00	36
44 21	4 10	29	52
44 43	5 59	1 00	43
45 13	7 24	1 32	41
46 21	9 26	2 30	51

Mean	=	44	43	39
δ	=	7	37	54
ϕ	=	52	21	33

3rd method—By two observed altitudes of a star, or the altitudes of two stars, and the elapsed time between the observations.

In addition to the latitude this method also serves to determine the time and azimuth.

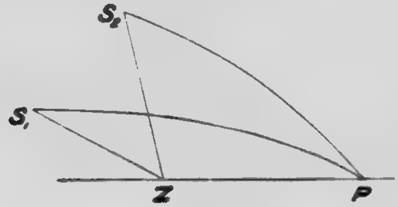


FIG. 19

Let S_1 and S_2 be the positions of the star or stars at the instants of observation. The first step in the reduction is to determine the difference of hour angle S_1PS_2. If the sun is observed twice, this angle is equal to the elapsed interval of apparent time between the observations, though usually the effect of the change in the equation of time may be neglected. If one fixed star has been observed the angle S_1PS_2 is equal to the elapsed sidereal interval between the observations. If two stars are observed at the times T_1 and T_2, the right ascensions being α_1 and α_2, then

$$S_1PS_2 = (\alpha_1 - \alpha_2) - (T_1 - T_2) \tag{62}$$

S_1 being the more easterly star. The interval $T_1 - T_2$ must be in sidereal time.

Then, PS_1 and PS_2 being known, the triangle S_1PS_2 may be solved, finding S_1S_2 and PS_1S_2. The three sides of the triangle ZS_1S_2 are now known, so that it may be solved, finding the angle ZS_1S_2. Then $PS_1Z = ZS_1S_2 - PS_1S_2$. The triangle PZS is finally solved, finding PZ the co-latitude. Completing the solution gives also the hour angle ZPS and the azimuth PZS.

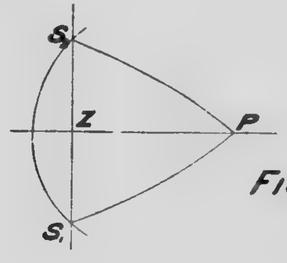


FIG. 20.

This method is further developed in works on navigation, in which graphical solutions are given.

4th method—By transits of stars across the prime vertical.

A star whose declination lies between the limits 0° and ϕ will cross the prime vertical above the horizon twice in its diurnal course.

The times of transit across the p.v. may be observed by means of a transit adjusted in the p.v. If S_1 and S_2 are the two positions of a star at the instant of observation, then the elapsed sidereal interval between the observations is equal to the angle S_1PS_2, and half that interval is the hour angle of the star at either observation. Transposing eq. (24) we have

$$\tan\phi = \frac{\tan\delta}{\cos\tau}; \tag{63}$$

by which the latitude may be found.

This method is little used with small instruments, but when applied to the astronomical transit instrument it is one of the most precise methods known for determining latitude.

5th method—By observations of stars at elongation.

If two circumpolar stars be selected, whose times of elongation, one east and the other west of the meridian, are not widely different, we have for the two stars, applying eq. (23)

$$\sin A_1 = \frac{\cos\delta_1}{\cos\phi} \qquad \sin A_2 = \frac{\cos\delta_2}{\cos\phi} \tag{64}$$

whence

$$\frac{\sin A_1}{\sin A_2} = \frac{\cos\delta_1}{\cos\delta_2}.$$

From this by composition and division

$$\frac{\sin A_1 + \sin A_2}{\sin A_1 - \sin A_2} = \frac{\cos\delta_1 + \cos\delta_2}{\cos\delta_1 - \cos\delta_2}$$

or

$$\frac{\tan\frac{1}{2}(A_1+A_2)}{\tan\frac{1}{2}(A_1-A_2)} = -\cot\tfrac{1}{2}(\delta_1+\delta_2)\cot\tfrac{1}{2}(\delta_1-\delta_2);$$

from which finally

$$\tan\tfrac{1}{2}(A_1-A_2) = -\tan\tfrac{1}{2}(A_1+A_2)\tan\tfrac{1}{2}(\delta_1+\delta_2)\tan\tfrac{1}{2}(\delta_1-\delta_2) \tag{65}$$

From this may be found the difference of the azimuths of the two stars when their sum is known. The sum of the azimuths may be observed by pointing the telescope of a transit to each star in turn, when at elongation, noting the

readings of the horizontal circle and taking their difference. From the sum and difference of A_1 and A_2 their separate values may be found. The latitude then follows by either equation

$$\cos\phi = \frac{\cos\delta_1}{\sin A_1} = \frac{\cos\delta_2}{\sin A_2} \tag{66}$$

This method was due to Prof. J. S. Corti.

The best stars for observation are those having large azimuths when at elongation, or whose declinations do not greatly exceed the latitude. Their elongations then occur at high altitudes, and therefore this principle must not be pushed to an extreme, as the effect of an unknown inclination error of the horizontal axis of the transit increases rapidly with the altitude.

5. Determination of Azimuth by Observation.

1st method—By meridian transits.

The time of meridian transit of any star may be computed as shewn on pp. 11 and 16. If the correction of a chronometer be known, the chronometer time of transit may be found. By directing the sight line of a well adjusted transit to the star at that instant, it will thus be placed in the meridian plane, and a meridian line may then be established on the ground; or by horizontal circle readings when pointing to the star and a mark, the azimuth of the latter may be determined.

It is clear that a slow-moving circumpolar star is best for this observation, as then the effect of an error in the computed time of transit is a minimum. The rate of change of azimuth of a star when crossing the meridian is given by the relation

$$\Delta A = 15.\Delta\tau \frac{\cos\delta}{\sin(\phi-\delta)} \tag{67}$$

(see eq. 75) ΔA being expressed in arc and $\Delta\tau$ in time. In the case of the pole star over 2^m are required for a change of azimuth of $1'$, when crossing the meridian.

2nd method—By transits across any vertical circle, the latitude being known.

Having computed the hour angle from the observed time, the data of the problem are τ, δ, and ϕ, and the azimuth of the star may be computed by means of (6) and (7), or

$$\tan\theta = \frac{\tan\delta}{\cos\tau}, \quad \tan A = \frac{\tan\tau\cos\theta}{\sin(\theta-\phi)}$$

The same considerations as in the last method lead to the choice of a close circumpolar star for this observation. The equation from which (6) was derived may be placed in more convenient forms. Thus it may be written,

$$\tan A = \frac{\sin\tau}{\cos\phi\tan\delta - \sin\phi\cos\tau};$$

then multiplying the right-hand member through by sec ϕ cot δ, this becomes

$$\tan A = \frac{\sec\phi\cot\delta\sin\tau}{1-\tan\phi\cot\delta\cos\tau} \tag{68}$$

This form is convenient when subtraction log's are available. (See *Manual of Survey of Dominion Lands.*)

Again, the above equation may be written

$$\tan A = \frac{\sin \tau}{\cos\phi \cot p - \sin\phi \cos\tau}$$

$$= \frac{\sin \tau}{\cos\phi \cot p(1-\tan p \tan\phi \cos\tau)}$$

$$= \frac{\sin\phi \tan p}{\cos\phi}(1-\tan p \tan\phi \cos\tau)^{-1}$$

Then expanding and neglecting powers of A and p above the second, we have

$$A = \frac{p \sin\tau}{\cos\phi}(1+p \tan\phi \cos\tau)$$

A and p are here expressed in circular measure. Writing them $A \sin 1''$ and $p \sin 1''$, in which they are now expressed in seconds, the equation becomes

$$A = \frac{p \sin\tau}{\cos\phi}(1+p \sin 1'' \tan\phi \cos\tau) \qquad (69$$

The omitted terms in the above expansions become important in high latitudes, but up to latitude 50°, in the case of the pole star, they will not exceed $2''$, and up to latitude 60° $4''.5$. They attain a maximum value when $\tau = 2^h$, about, and vanish when τ slightly exceeds 4^h.

In taking the observation the procedure is as follows:

Point to the reference point and note H.C.R.

Then point to the star, note time and H.C.R.

Then reverse instrument and again point to the star and note time and H.C.R.

Then point to the reference point and note H.C.R.

The means of the H.C.R's on the star and reference point are then taken, increasing or diminishing one in each case by 180°; and also the mean of the times of pointing to the star, from which the hour angle is derived.

Having computed the azimuth by (68) or (69), let:

A_s denote the azimuth of the star reckoned from the north in the direction *ESW*;

A_p that of the reference point.

R_s the H.C.R. on pointing to the star

R_p that on pointing to the reference point.

Then $\qquad A_p - A_s = R_p - R_s$

or $\qquad A_p = A_s + R_p - R_s \qquad (70)$

Example.—The foll 'ng observations were taken in Aug., 1904, at a place ,, latitude 46° 54′:

Pl. obs'd.	*Cir.*	*H.C.R.*	*Watch*
R.P.	*R.*	178° 14′.5	
Polaris	*R.*	57 .5	$15^h\ 55^m\ 08^s$
Polaris	*L.*	181 02 .5	16 01 05
R.P.	*L.*	358 14 .5	

The watch correction was found by observing the meridian transit of α Scorpii, as follows:

Watch time of transit	$= 16^h\ 23^m\ 00^s$
R't ascension of star	$= 16\ \ 23\ \ 34$
Watch corr'n	$= \quad +34$

From the N.A.

$$\alpha \text{ (of Polaris)} = 1^h\ 25^m\ 03^s$$
$$\delta = 88°\ 47'\ 28''$$
$$\therefore\ p = 4352''$$

The computation then proceeds as follows:

	Mean of obs'd times	$= 15^h\ 58^m\ 06^s$
	Watch corr'n	$= \quad +34$
Eq. (40)	Sid. time	$= 15\ \ 58\ \ 40$
	α	$= 1\ \ 25\ \ 03$
	t	$= 14\ \ 33\ \ 37$
	τ	$= 9\ \ 26\ \ 23$
		$= 141°\ 35'\ 45''$

Eq. (68)

$\log \sec \phi = 10.165405$	$\log \tan \phi = 10.028825$
$\log \cot \delta = 8.324328$	$\log \cot \delta = 8.324328$
$\log \sin \tau = 9.793235$	$\log \cos \tau = 9.894122n$
8.282968	8.2472
Subt. $\log = 0.007610$	

$$\log \tan A = 8.275358$$
$$A = 1°\ 04'\ 48''$$

Eq. (70)

$$A_s = 1°\ 04'\ 48''$$
$$R_p = 178\ \ 14\ \ 30$$
$$179\ \ 19\ \ 18$$
$$R_s = 1\ \ 00\ \ 00$$
$$A_p = 178°\ 19'\ 18''$$

The computation by (69) is as follows:

log p	$= 3.638689$
log sin τ	$= 9.793235$
log cos ϕ	$= 9.834595$
	3.431924
log 1st term	$= 3.597329$
log p	$= 3.638689$
log sin 1″	$= 6.685575$
log tan ϕ	$= 10.028825$
log cos τ	$= 9.984122n$
log 2nd term	$= 1.844540n$
1st term	$= 1° \; 05' \; 57''$
2nd term	$= \quad -1 \;\; 10$
A	$= 1 \;\; 04 \;\; 47$

This method may be used to advantage in finding the variation of a compass. An explorer's instrumental equipment may consist of a sextant and a compass. With the former instrument an observation of the sun for time may be taken. If the compass bearing of the sun's limb then be taken, the true azimuth of that body may be computed in terms of τ δ and ϕ, which, compared with the magnetic azimuth, will give the variation. The quantity S sec h must be added to or subtracted from the azimuth of the sun's centre to obtain the azimuth of the limb. h is given by (5) and need only be known approximately.

The equations then are:

$$\tan\theta = \frac{\tan\delta}{\cos\tau} \qquad \sin h = \frac{\sin\delta\cos(\theta-\phi)}{\sin\theta}$$

$$\tan A = \frac{\tan\tau\cos\theta}{\sin(\theta-\phi)} \qquad \Delta A = S\sec h$$

The best time for this observation is when the sun is near the prime vertical.

3rd method—By an observed altitude.

The method of observation is described on *p.* 65 et seq.

The data are h δ and ϕ, and the reduction is made by one of the equations (8), (9) or (10).

Example.—The following observations of the sun for azimuth and time were taken on July 30, 1914, at a place in latitude $44°24'09''$, and approximate longitude $5^h\,18^m 15^s$ W.:

Pt. obs'd.	*Cir.*	*H.C.R.*	*V.C.R.*	*Watch*
R.P.	*R.*	23° 26′.5		
$\odot$	*R.*	219 22	28° 26′.5	$4^h\,56^m\,39^s$ p.m
$\odot$	*L.*	40 34	27 23 .5	59 47.5
R.P.	*L.*	203 26		

The reduction is as follows:

To find the azimuth:

Mean of V.C.R's.	$=27°$	$55'$	$00''$
r	=	1	49
	27	53	11
p	=		8
h	=27	53	19
ζ	=62	06	41
ϕ	=44	24	09
δ	=18	34	07
s'	=62	32	28
$s'-\phi$	=18	08	19
$s'-\delta$	=43	58	21
$s'-\zeta$	= 0	25	47

log cos s' = 9.663807
log sin$(s'-\delta)$ = 9.841555

log cos$(s'-\zeta)$ = 9.999988
log sin$(s'-\phi)$ = 9.493203

9.505362
9.493191

log tan² $\frac{1}{2}A$	= 10.012171		
log tan $\frac{1}{2}A$	= 10.006085		
$\frac{1}{2}A$	= 45°	24′	05″
A	= 90	48	10
A_s	= 269	11	50
R_p	= 23	26	15
	292	38	05
R_s	= 219	58	
A_p	= 72	40	05

To find the time:

log sin$(s'-\phi)$ = 9.493203
log sin$(s'-\delta)$ = 9.841555

log cos s' = 9.663807
log cos$(s'-\zeta)$ = 9.999988

9.334758
9.663795

log tan² $\frac{1}{2}\tau$	= 9.670963			
log tan $\frac{1}{2}\tau$	= 9.835481			
$\frac{1}{2}\tau$	= 34°	23′	54″	
τ	= 68	47	48	
A.T.	$=4^h$	35^m	11^s	.2
E	= +6	16		.0
M.T.	= 4	41	27	.2
		18	15	
Stand. *T*	= 4	59	42	.2
Watch	= 4	58	13	.2
ΔT	= +1	29		.0

Example.—The following observations were taken with a small transit in Sept., 1899, to determine azimuth, time and latitude.

Pt. obs'd.	Cir.	H.C.R.	V.C.R.	Watch
Arcturus	*R*	157° 10′	27° 21′	$7^h\,43^m\,11^s$ p.m.
Arcturus	*L*	337 58	27 06	46 23
Arcturus	*R*	158 42	26 46 .5	49 35
Arcturus	*L*	339 37	26 19	53 24
Arcturus	*R*	160 22	25 59 .5	56 43
Arcturus	*L*	341 16	25 33	8 00 29
R.P.	*R*	225 45		
Altair	*L*		34 23	16 55
Altair	*R*		34 25	19 11
Altair	*L*		34 27 .5	21 18
Altair	*R*		34 29	23 00
Altair	*L*		34 30	24 45
Altair	*R*		34 32	27 19

A mean time watch was used. Arcturus was to the west of the meridian, and Altair near the meridian and east of it.

The approximate meridian altitude of Altair was observed to be

$$34°\ 39'\ 30''$$

whence a value of the latitude for reducing the azimuth observations was found as follows:

$$\begin{aligned} h' &= 34°\ 39'\ 30'' \\ r &= 1\ \ 23 \\ \hline h &= 34\ \ 38\ \ 07 \\ \zeta &= 55\ \ 21\ \ 53 \\ \delta &= +8\ \ 36\ \ 23 \\ \hline \phi &= 63\ \ 58\ \ 16 \end{aligned}$$

The apparent places of the two stars were:

	α	δ
Arcturus...........	$14^h\,11^m\,05^s$	+19° 42′ 24″
Altair.............	19 45 55	+ 8 36 23

The reduction of the first azimuth observation is as follows:

V.C.R., Cir. *R*	=27° 27′
V.C.R., Cir. *L*	=27 06
Mean	=27 16 30″
r	= 1 51
h	=27 14 39
ζ	=62 45 21

Eq. (10)	s'	$= 73° \; 13' \; 00''.5$
	$s' - \phi$	$= 9 \; 14 \; 44 \; .5$
	$s' - \delta$	$= 53 \; 30 \; 36 \; .5$
	$s' - \zeta$	$= 10 \; 27 \; 39 \; .5$
	$\log \cos s'$	$= 9.460524$
	$\log \sin(s' - \delta)$	$= 9.905235$
	$\log \cos(s' - \zeta)$	$= 9.992721$
	$\log \sin(s' - \phi)$	$= 9.205930$
		9.365759
		9.198651
	$\log \tan^2 \frac{1}{2}A$	$= 10.167108$
	$\log \tan \frac{1}{2}A$	$= 10.083554$
	$\frac{1}{2}A$	$= 50° \; 28' \; 40''.5$
	A	$= 100 \; 57 \; 21 \; .0$
Eq. (70)	A_s	$= 259 \; 02 \; 39$
	R_p	$= 225 \; 45$
		$484 \; 47 \; 39$
	R_s	$= 157 \; 34$
	A_p	$= 327 \; 13 \; 39$

Reducing the remaining azimuth observations in a similar manner, and taking the mean, the result is

$$A_p = 327° \; 13' \; 31''$$

In order to reduce the latitude observations it is necessary to find the hour angle of Altair corresponding to each of the observed times. This may be done by computing the hour angle of Arcturus from the observations of that star, and combining it with the difference of right ascension of the two stars. Thus:

Eq. (13)	$\log \sin(s' - \phi)$	$= 9.205930$
	$\log \sin(s' - \delta)$	$= 9.905235$
	$\log \cos s'$	$= 9.460524$
	$\log \cos(s' - \zeta)$	$= 9.992721$
		9.111165
		9.453245
	$\log \tan^2 \frac{1}{2}\tau$	$= 9.657920$

$\log \tan \frac{1}{2}\tau$		$=9.828960$
$\frac{1}{2}\tau$		$=33° 59' 54''$
τ		$=67\ 59\ 48$
		$=4^h\ 31^m 59^s.2$

Reducing the remaining observations in the same way, the hour angles are:

$$
\begin{array}{rrrr}
 & 4^h & 31^m & 59^s.2 \\
 & & 38 & 45\ .1 \\
 & & 45 & 54\ .4 \\
\hline
\text{Mean} = & 4 & 38 & 52\ .9
\end{array}
$$

The difference of r.a. of the two stars is

$$5^h\,34^m\,50^s\ ;$$

therefore the hour angle of Altair

$$
\begin{array}{rrrr}
= & 4^h & 38^m & 53^s \\
 & -5 & 34 & 50 \\
\hline
= & & -55 & 57
\end{array}
$$

(the star being east of the meridian) at an instant equal to the mean of the observed times, or

$$7^h\,51^m\,37^s.5$$

Then as the change of hour angle of a star is equal to the change in the sidereal time, the hour angle of Altair at the time of the first latitude observation is found as follows:

Observed time, 1st obs'n	=	$8^h\ 16^m\ 55^s$
Mean of times of az. obs'ns	=	7 51 37 .5
Diff.	=	25 17 .5
Equivalent sid. interval	=	25 21 .7
Hour angle at mean of times	=	−55 57
Hour angle at 1st lat. obs'n.	=	−30 35

The hour angles of Altair are thus found to be

$$
\begin{array}{rr}
-30^m & 35^s \\
28 & 19 \\
26 & 12 \\
24 & 29 \\
22 & 44 \\
20 & 10
\end{array}
$$

The latitude observations are now reduced as follows:

Eq. (61)	h'	$=34° 23' 00''$
	r	$=\ \ 1\ 24$
	h	$=34\ 21\ 36$

ζ	$= 55 \quad 38 \quad 24$
ϕ	$= 63 \quad 58 \quad 16$
δ	$= \ \ 8 \quad 36 \quad 23$
h_o	$= 34 \quad 38 \quad 07$
$\log \cos \phi$	$= 9.642291$
$\log \cos \delta$	$= 9.995082$
$\log \cos h_o$	$= 9.915287$
	9.637373
	9.722086
$\log m$	$= 3.263353$
$\log y$	$= 2.985439$
$y = 967''$	$= \ 0° \ 16' \ 07''$
ζ	$= 55 \quad 38 \quad 24$
ζ_o	$= 55 \quad 22 \quad 17$
δ	$= \ \ 8 \quad 36 \quad 23$
ϕ	$= 63 \quad 58 \quad 40$

The complete latitude results are:

$\phi = 63° \ 58' \ 40''$
59
28
28
53
46

Mean $= 63 \quad 58 \quad 42$

The inclusion of the small term in the expression for ϕ increases this result by less than $1''$.

The effect of the error of $26''$ in the value of the latitude used in the computation of A is found by the formula

$$dA = - \frac{d\phi}{\cos \phi \tan \tau}$$

to be about $21''.5$.

4th method—By an observation of a circumpolar star at elongation.

The azimuth and hour angle of the star may be found by (22) and (23). From the former the time of elongation may be computed.

Description of method of taking the observation.

In the case of the pole star, assuming $\alpha = 1^h\,26^m$, $\delta = 88°\,50'$, we find $\tau = 5^h\,58^m\,20^s$, and $\therefore\ \Theta = 7^h\,21^m\,20^s$, the sidereal time of western elongation. This may be used to compute approximately the time of either elongation at any time of the year.

5th method—By transits of stars across the vertical circle of Polaris.

From the observed times of transit of two stars across the same vertical circle, the azimuth of that circle may be computed.

FIG. 21

To find the azimuth: In Fig. 21, S_1 is the position of Polaris at the time of transit and S that of an equatorial star. SZS_1 is then the vertical circle of the instrument, and PZ the meridian. The angle SPS_1 (denoted by Δ) differs from the difference of r.a. of the two stars by the sidereal interval between their transits, or

$$\Delta = (\alpha_1 - \alpha) - (T_1 - T) \tag{71}$$

T_1 and T being the observed times of transit of Polaris and the other star, respectively, α_1 and α their right ascensions. In computing Δ the subtractions should be algebraic; Δ will then be affected by the + sign if the star S is west of the meridian, and by the − sign if east.

We next take the equations:

$$\sin\Delta\cot C = \cos\delta\tan\delta_1 - \sin\delta\cos\Delta$$
$$\sin\tau\cot C = \cos\delta\tan\phi - \sin\delta\cos\tau$$
$$\sin A = \frac{\cos\delta\sin C}{\cos\phi}$$

which are obtained from (5) and (3), *Sph. Trig.* From the first of these we have

$$\tan C = \frac{\sin\Delta}{\cos\delta\cot p - \sin\delta\cos\Delta},$$
$$= \frac{\sin\Delta}{\cos\delta\cot p(1 - \tan p\tan\delta\cos\Delta)},$$

$$= \frac{\sin \Delta}{\cos \delta} \tan p(1+\tan p \tan \delta \cos \Delta +),$$

$$= \frac{p \sin \Delta}{\cos \delta} (1+p \tan \delta \cos \Delta) \qquad (72)$$

neglecting p^3. Again, from the second equation we have

$$\sin \tau \frac{\cot C}{\sin \delta} + \cos \tau = \frac{\tan \phi}{\tan \delta},$$

or
$$\tau \frac{\cot C}{\sin \delta} + 1 - \frac{\tau^2}{2} = \frac{\tan \phi}{\tan \delta},$$

again neglecting the cube and higher powers of small quantities; $\therefore$

$$\tau \frac{\cot C}{\sin \delta} - \frac{\tau^2}{2} = \frac{\tan \phi}{\tan \delta} - 1 = \frac{\sin (\phi - \delta)}{\cos \phi \sin \delta}.$$

Then assuming as a first approximation

$$\tau \frac{\cot C}{\sin \delta} = \frac{\sin (\phi - \delta)}{\cos \phi \sin \delta},$$

or
$$\tau = \frac{\sin (\phi - \delta)}{\cos \phi} \tan C,$$

we have by substitution for τ^2 in the above equation

$$\tau = \frac{\sin (\phi - \delta)}{\cos \phi} \tan C +$$

$$= \frac{p \sin (\phi - \delta) \sin \Delta}{\cos \phi \cos \delta} (1+p \tan \delta \cos \Delta)$$

by (72), o seconds of arc

$$\tau = \frac{p \sin(\phi - \delta) \sin \Delta}{\cos \phi \cos \delta} (1+p \sin 1'' \tan \delta \cos \Delta) \qquad (73)$$

If the time star be observed below the pole, then δ changes its sign, and τ becomes the hour angle reckoned from lower culmi ıation.

To find the azimuth we have from the third of the above equations

$$A = C \frac{\cos \delta}{\cos \phi}$$

or by (72)
$$A = \frac{p \sin \Delta}{\cos \phi} (1+p \sin 1'' \tan \delta \cos \Delta) \qquad (74)$$

A and p being in seconds of arc.

Comparing equations (73) and (74) we see that

$$\tau = A \frac{\sin (\phi - \delta)}{\cos \delta} \qquad (75)$$

Example.—The following observations were taken at Toronto, Mar. 29, 1899:

Pt. obs'd.	*H.C.R.*	*Watch*
R.P.	45° 18′	
Polaris	73 33 .5	$8^h 30^m 51^s$
ζ Hydræ	73 33 .5	8 34 43

The apparent places of the stars were:

	α	δ
Polaris......	$1^h 21^m 21^s$	+88° 46′ 23″
ζ Hydræ.....	8 50 06	+ 6 19 35

We have then the following data:

$\Delta = 111° 13'$ (Eq. 71.)
$\phi = 43\ 39\ 36''$
$\delta = 6\ 19\ 35$
$p = 4417''$;

so that the computation proceeds as follows:

Eq. (74)	log sin δ	=9.96952
	log p	=3.64513
	log cos ϕ	=9.85941
		3.61465
	log 5692	=3.75524
	log tan δ	=9.04480
	log cos Δ	=9.55858n
	log p	=3.64513
	log sin 1″	=$\bar{6}$.68557
	log −5	=0.68932n
	$\therefore A = 5687''$	= 1° 34′ 47″
Eq. (70)	A_s	=358° 25′ 13″
	R_p	= 45 18
		403 43 13
	R_s	= 73 33 30
	A_p	=330° 09′ 43″
Eq. (75)	log A	=3.75488
	log sin$(\phi-\delta)$	=9.78280
	log cos δ	=9.99735
		3.53768
	log 3470	=3.54033

$\therefore\ \tau = 3470'' = 231^s$	$= 3^m\ 51^s$
α(ζ Hydræ)	= 8 50 06
Θ	= 8 46 15
L	= 5 17 35
Θ(at Gr.)	= 14 03 50
Equiv. M.T. int'l	= 14 01 32
M.T. of sid. noon	= 23 33 23
	37 34 55
	13 34 55
Standard Time	= 8 34 55
Watch	= 8 34 43
Watch corr'n	= +12

6th method—By the observed angular distance of the sun from a terrestrial point.

This method is useful when the sextant is the only instrument available.

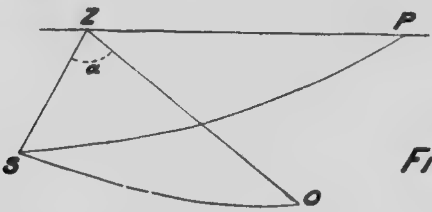


FIG. 22

In Fig. 22 S is the centre of the sun, and O the terrestrial point. The observation comprises:

Measuring the angular distance SO,
Noting the time of observation, and
Measuring the altitude of O.

The latitude being known, the altitude and azimuth of the sun's centre are computed by (4), (5) and (6). The apparent altitude is then found by subtracting the parallax and adding the refraction The measured angular distance is corrected for semi-diameter. We have then

$$\tan^2 \tfrac{1}{2}\alpha = \frac{\sin(s-ZS)\sin(s-ZO)}{\sin s \sin(s-SO)}$$

in which $$s = \frac{ZS+ZO+SO}{2}$$

If then, h' = the apparent altitude of the sun

H = the altitude of O

D = the angular distance SO

we find on substituting

$$s' = \frac{h'+H+D}{2}$$

$$\tan^2 \tfrac{1}{2} a = \frac{\sin(s'-H)\sin(s'-h')}{\cos s' \cos(s'-D)} \qquad (76)$$

If H is so small that it may be neglected, as is often the case in hydrographic surveys, then (76) becomes

$$\tan^2 \tfrac{1}{2}a = \tan \tfrac{1}{2}(D \times h') \tan \tfrac{1}{2}(D-h') \qquad (77)$$

The azimuth of O then is

$$A \pm a$$

If the correction of the watch is not known the observer may proceed as follows:

Measure the altitude of the sun, then the angular distance SO, then again the altitude of the sun, noting the watch time of each of the three measurements. The altitude of the sun at the instant of measuring SO may then be interpolated. The altitude of O is measured as before. A may then be computed from the data h δ and ϕ by either (8), (9) or (10). The remainder of the reduction is as before.

6. Determination of Longitude by Observation.

The engineer is seldom called upon to determine longitude, so that only some methods useful to the explorer will be here described, and also in outline the most precise method known, viz., that by the electric telegraph.

The difference of longitude between two places may be defined as the angle between the planes of their meridians.

It was seen—p. 14—that the local times of two places differ by an amount equal to their difference of longitude, expressed in time. Any method, therefore, that serves to compare the local times of the two places, at the same absolute instant of time, will determine their difference of longitude.

1st method—By portable chronometers.

If the correction of a chronometer on the local time of a place A is found by observation, and also its rate, and the chronometer is then transported to another place B, and its correction on the local time of that place found, the local times of the two places may be thus compared: Let

ΔT, δT = the correction and rate found at A at the time T;

$\Delta T'$ = the correction found at B at the time T' $(=T+t)$

Then at the instant T' the true time

$$\text{at } A = T+t+\Delta T+t\,.\,\delta T,$$
$$\text{at } B = T+t+\Delta T';$$

the difference of which is

$$\Delta L = \Delta T + t\,.\,\delta T - \Delta T',$$

or the difference of the corrections of the chronometer on the times of the two places at an assumed instant of time.

2nd method—By signals.

Any signal that may be seen at the two places may be used to compare their local times. A chain of observing stations may be established between the extreme stations, with inter-

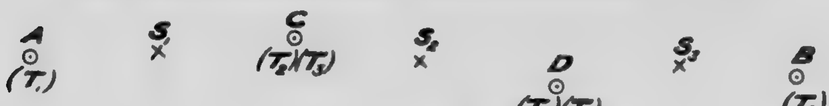


Fig. 23

mediate signal stations, so that the method may be used between points at a considerable distance apart. The signal used may be the disappearance of a light, a flash of gunpowder, etc.

Let A and B be the terminal stations, C and D intermediate stations, and S_1 S_2 and S_3, signal stations (Fig. 23). Then if a signal be made at S_1 which is perceived at A at the time T_1

and at C at the time T_2; and if then a signal be made at S_2 which is perceived at C at the time T_3 and at D at the time D_4, etc.; then, A being the more easterly station, we have

$$\Delta L = (T_1 - T_2) + (T_3 - T_4) + (T_5 - T_6)$$
$$= T_1 - (T_2 - T_3) - (T_4 - T_5) - T_6;$$

which shews that it is not necessary to know the corrections of the chronometers at the intermediate stations, but only their rates. The times T_1 and T_6 are the true local times at A and B, respectively.

Eclipses of Jupiter's satellites are also used in longitude determinations. As the satellite appears to fade out gradually the observed time of an eclipse will depend upon the power of the telescope used. But for this objection this method would be a useful one for finding longitude.

Reference may be made to the ephemeris.

3rd method—By the electric telegraph.

The observer at each station must be provided with a transit instrument, chronometer, and electro-chronograph, for determining time with precision, and also a portable switchboard by which connections can be made with the main telegraph line for sending signals to the other station.

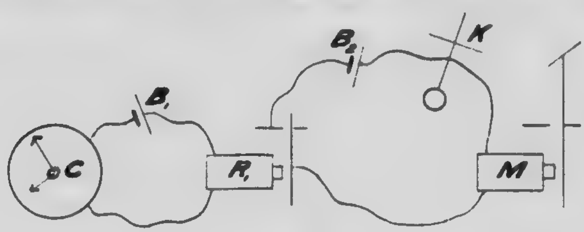


Fig. 24

The connections for observing the transits of stars in determining time are shewn in diagram in Fig. 24, and for sending arbitrary signals in Fig. 25.

The procedure at each station is to observe a set of stars for determining time and the instrumental constants. Then a series of signals is sent to the distant station, which are also recorded on the local chronograph. A second set of stars is then observed. By means then of the two time sets the correction of the chronometer on local time at the epoch of the signals can be interpolated.

These operations may be repeated on as many mutually clear nights at the two stations as may be considered necessary say five nights.

In Figs. 24 and 25—

C is the chronometer,
B_1 the chronometer battery,
R_1 the chronometer relay,
B_2 the chronograph battery,
M the chronograph magnet,
K the transit key.

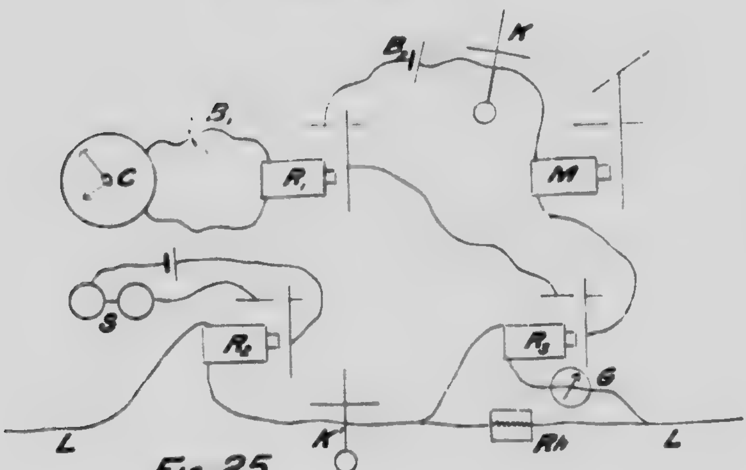


FIG. 25

Also in Fig. 25—

LL is the main line,
R_2 the sounder relay,
S the sounder,
R_3 the signal relay,
Rh a rheostat,
G a galvanometer,
K' the telegraph and signal key.

A signal is made by breaking the main line circuit by means of the signal key, which may be a special break-circuit key.

If now at a time T_1 at station A a signal is made which is recorded at B at the time T_1'; and if ΔT_1 $\Delta T_1'$ are the chronometer corrections on local time at the two stations, and x the time of transmission of the signal; then the difference of longitude is:

$$\Delta L = (T_1 + \Delta T_1) - (T_1' + \Delta T_1' - x)$$
$$= \Delta L_1 + x$$

in which $\Delta L_1 = (T_1 + \Delta T_1) - (T_1' + \Delta T_1')$

If a signal now be made at B at the time T_2', and recorded at A at the time T_2; then

$$\Delta L = (T_2 + \Delta T_2 - x) - (T_2' + \Delta T_2')$$
$$= \Delta L_2 - x$$

in which $\Delta L_2 = (T_2 + \Delta T_2) - (T_2' + \Delta T_2')$

Taking the mean of these values of ΔL x is eliminated, and we have

$$\Delta L = \frac{\Delta L_1 + \Delta L_2}{2}$$

4th method.—By moon culminations.

An examination of the moon's hourly ephemeris contained in the N. A. will show that the motion of that body in right ascension is very rapid. If then a value of that co-ordinate be found by observation, and the corresponding Gr. time be interpolated from the ephemeris, the error in the time due to the error in the observed quantity will not be excessive. The Gr. time being thus found at the instant of the observation, which also serves to determine the local time, the longitude follows by taking the difference of the two times.

To determine the moon's r.a. the meridian transit of the moon's limb and that of some neighbouring star are observed. Then let

θ and θ' = the sidereal times of transit of the moon's centre and a star.

α and α' = their right ascensions

and we have

$$\alpha - \alpha' = \theta - \theta'$$

or

$$\alpha = \alpha' + \theta - \theta'$$

which gives the moon's right ascension.

To find the sidereal time of the semi-diameter passing the meridian in order to correct the observed time of transit of the limb, let

σ = the sid. time of the S.D. passing the meridian

S = the moon's angular S.D.

$\Delta\alpha$ = the increase of the moon's r.a. in 1^m of M.T.

then $\dfrac{\Delta\alpha}{60.164}$ = the increase of the moon's r.a. in 1 sid. second;

and

$\sigma \dfrac{\Delta\alpha}{60.164}$ = its increase in the interval σ;

and $\therefore$

$$\sigma - \sigma \frac{\Delta\alpha}{60.164} = \frac{S \sec \delta}{15}$$

as each side of the equation expresses the time of S.D. passing the meridian if there were no change of r.a.; $\therefore$

$$= \frac{S}{15 \cos\delta\left(1 - \frac{\Delta\alpha}{60.164}\right)}$$

$$= \frac{60.164\, S}{15 \cos\delta\,(60.164 - \Delta\alpha)}$$

This quantity is given in the N.A.

To interpolate the Gr. M.T. corresponding to an observed value of the moon's r.a., let

α_0 = the ephemeris value nearest to α,
T_0 = the corresponding Gr. M.T.,
T = the Gr. M.T. corresponding to α,
x = $T - T_0$ (in seconds),
$\Delta\alpha$ = the increase of α in 1 minute of M.T. at the time T_0,
$\delta\alpha$ = the increase of $\Delta\alpha$ in 1 hour.

Then the increase of $\Delta\alpha$ in the interval x is

$$\frac{x}{3600} \cdot \delta\alpha;$$

$\therefore$ the value of $\Delta\alpha$ at the middle instant of the interval x is

$$\Delta\alpha + \frac{x}{7200}\,\delta\alpha$$

and $\therefore$ the increase of α in the interval x is

$$\frac{x}{60}\left(\Delta\alpha + \frac{x}{7200}\,\delta\alpha\right),$$

and $\therefore$ $$\alpha = \alpha_0 + \frac{x}{60}\left(\Delta\alpha + \frac{x}{7200}\,\delta\alpha\right),$$

Then $$x = \frac{60\,(\alpha - \alpha_0)}{\Delta\alpha + \frac{x}{7200}\,\delta\alpha} = \frac{60\,(\alpha - \alpha_0)}{\Delta\alpha\left(1 + \frac{x}{7200}\,\frac{\delta\alpha}{\Delta\alpha}\right)},$$

$$= \frac{60\,(\alpha - \alpha_0)}{\Delta\alpha}\left(1 - \frac{x}{7200}\,\frac{\delta\alpha}{\Delta\alpha}\right), \text{ nearly}$$

$$= x' - \frac{x'^2}{7200}\,\frac{\delta\alpha}{\Delta\alpha} \quad \text{nearly}$$

in which $x' = \dfrac{60\,(\alpha - \alpha_0)}{\Delta\alpha}$

Then $T = T_0 + x$

If then Θ is the Gr. sid. time corresponding to T we have

$$L = \Theta - \alpha$$

A more accurate method than the foregoing is to take observations for determining α on the same night at the

station whose longitude is required and also at another station whose longitude has been well determined. Thus the increase in a while the moon is passing over the interval between the two meridians is determined. This increase, divided by the increase in 1 hour of longitude, gives the difference of longitude in hours. Thus if

a_1 and a_2 = the values of a found at the two stations,

H = the increase of a in 1 hour of longitude while passing over the interval between the two meridians;

then
$$\Delta L = \frac{a_2 - a_1}{H}$$

H may be taken from the ephemeris.

7. The Theodolite and the Sextant.

The Theodolite.

For a knowledge of the construction and method of adjustment of the engineer's transit theodolite reference may be made to any standard work on surveying.

A well constructed and adjusted transit should fulfil the following conditions:

(1) The vertical and horizontal axes should pass through the centres of the horizontal and vertical circles, respectively, and should be perpendicular to their planes.

(2) The axis of the alidade of the horizontal circle should coincide with the axis of the circle.

(3) The line joining the zeros of the verniers of either circle (assuming that each is read by two verniers) should pass through the centre of the circle.

(4) The extreme divisions of each vernier should coincide at the same time with divisions of its circle.

(5) The horizontal axis should be perpendicular to and intersect the vertical axis.

(6) The sight line of the telescope should be perpendicular to and intersect the horizontal axis, and in all positions of the focusing slide. It should also intersect the vertical axis.

(7) The two threads in the telescope, whose intersection determines a point on the sight line, should be truly horizontal and vertical, respectively, when the instrument is adjusted for observation.

(8) The levels attached to the horizontal plate should read zero when the vertical axis is plumb.

(9) When either vernier of the vertical circle reads zero, and also the level attached to the alidade of that circle, the sight line should be horizontal.

Conditions 1, 2, 3, 4 and the second part of 6 are fulfilled by the maker in the construction of the instrument; the others, and sometimes 3, can be attended to by the observer. With regard to 9, the alidade of the vertical circle of a transit intended for astronomical observation should be provided with a level capable of detecting a change of inclination considerably smaller than the least count of the vernier. The position of the alidade should be adjustable by means of a slow-motion screw, so that the bubble of its level may readily be brought to the centre, after plumbing the vertical axis of the instrument.

It is proposed to examine the effects of these errors of construction and adjustment, shewing how in most cases they may be eliminated.

(1) The effect of an inclination of the horizontal axis.

In Fig. 26, which is a projection of the celestial sphere on the plane of the horizon, the horizontal rotation axis of the transit is assumed to be inclined at a small angle to the horizon, so that the collimation axis—defined as a right line through the optical centre of the objective perpendicular to the horizontal axis—traces on the celestial sphere the great circle $A'PZ'$. P being any point and APZ a vertical circle,

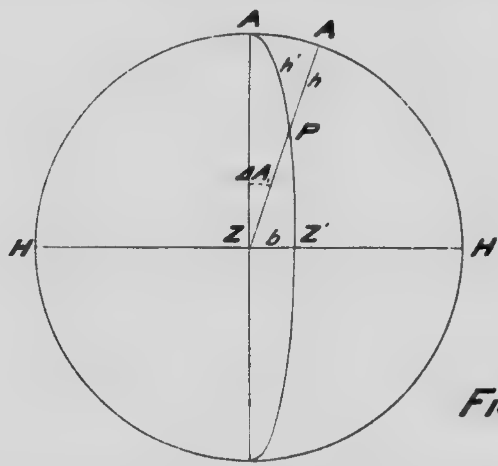


FIG. 26

the true altitude of P is the arc AP; and the apparent altitude, affected by the inclination of the axis, the arc $A'P$. Z' is the zenith of the instrument, and ZZ' is equal to the inclination b. It is clear that the effect of b on the H.C.R. is shewn by the spherical angle AZA'. To find an expression for this angle we have in the triangle PZZ'

$$\tan PZZ' = \frac{\tan PZ'}{\sin ZZ'};$$

or
$$\cot \Delta A_1 = \frac{\cot h'}{\sin b};$$

or
$$\tan \Delta A' = \tan h' \sin b;$$

or, as ΔA_1 and b are small, we may write this

$$\Delta A_1 = b \tan h', \tag{79}$$

or the effect of an inclination of the horizontal axis on the H.C.R. varies as the tangent of the altitude of the point sighted.

In measuring the horizontal angle between two points it is evident that the effect of b is nil if the altitudes of the two

points are equal, and that it increases with the difference of the altitudes. A reversal of the instrument reverses the algebraic sign of ΔA_1, so that its effect on a horizontal angle is eliminated by the reversal.

To find the effect of b on the measurement of a vertical angle we again refer to the triangle PZZ', from which we have

$$\cos PZ = \cos PZ' \cos ZZ'$$

or

$$\sin h = \sin h' \cos b$$

Then denoting $h'-h$ by Δh and expanding $\cos b$ we have

$$\sin(h'-\Delta h) = \sin h' \left(1 - \frac{b^2}{2}\right)$$

or

$$\sin h' - \Delta h \cos h' = \sin h' - \frac{b^2}{2} \sin h'$$

by expanding the sin and cos of Δh and neglecting its square and higher powers; $\therefore$

$$\Delta h = \frac{b^2}{2} \tan h'$$

It appears then that the effect of b on a vertical angle varies as the square of b. Introducing the values of Δh and b in seconds we have

$$\Delta h = \frac{\sin 1''}{2} b^2 \tan h \tag{80}$$

This is a very small quantity; for, assuming $b = 1'$ and $h' = 45°$, we find $\Delta h = 0''.0087$; it may therefore be safely neglected. It is not eliminated by reversal.

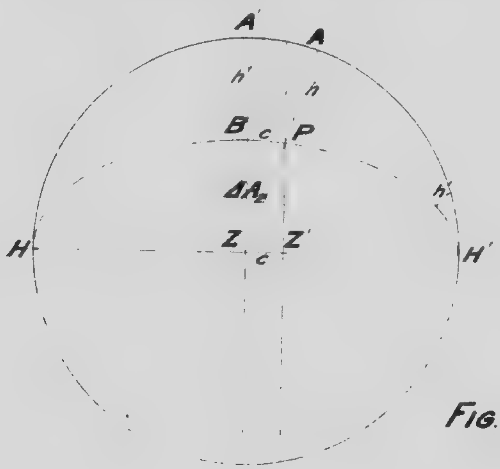


FIG. 27

5—

(2) The effect of a collimation error; *i.e.*, an error arising from non-coincidence of the sight line and the collimation axis as above defined.

Assuming that there is no inclination error the sight line in this case will trace on the celestial sphere a small circle parallel to the great circle traced out by the collination axis. In Fig. 27 PZ' is the small circle, and $A'BZ$ the great circle traced out by the collimation axis. H and H' are the poles of those circles, Z' being the zenith of the instrument; ZZ' or PB is the collimation error, denoted by c.

To find the effect of c on a H.C.R., denoted by ΔA_2, we have in the triangle BZP

$$\tan BZP = \frac{\tan BP}{\sin BZ}$$

or $$\tan \Delta A_2 = \frac{\tan c}{\cos h'}$$

or very nearly $$\Delta A_2 = c \sec h' \tag{81}$$

or the effect of a collimation error on a H.C.R. varies as the secant of the altitude of the point sighted.

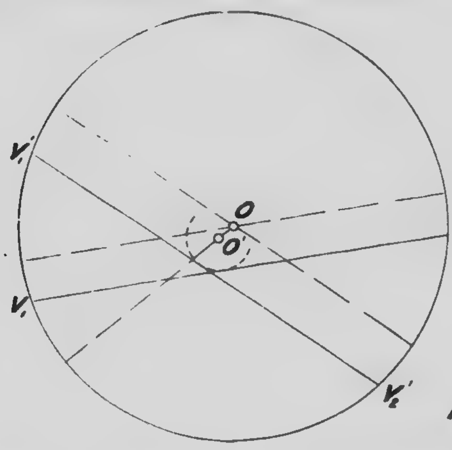


FIG. 28

The effect of this error on the measurement of a horizontal angle evidently also increases with the difference of the altitudes of the two points sighted, and is eliminated by a reversal of the instrument.

To find the effect of c on the measurement of a vertical angle we have in the triangle BZP

$$\cos PZ = \cos BZ \cos BP$$

or $$\sin h = \sin h' \cos c$$

As this is the same equation as was derived in the discussion

of the last error, it follows that equation (80) also expresses the error in this case.

(3) To find the effect of a non-fulfilment of condition 1, 2 or 3, so that the line joining the zeros of the two verniers does not pass through the centre of the circle.

The circle in Fig. 28 represents the graduated circle, of which O is the centre. O' is the centre of the alidade. Also the line joining the zeros of the two verniers does not pass through the point O'. It is clear from the figure that if in any position of the alidade the reading of the vernier V_1 is less than what it would be if the line V_1V_2 occupied a parallel position passing through O, then the reading of V_2 will be in excess by the same amount. By taking the mean of the two values of an angle, found by taking readings of both verniers, the effect of eccentricity is therefore eliminated.

By a different process it may be shewn that the effect of eccentricity may be eliminated by any number of equidistant verniers.

With regard to condition 8, it is convenient that the plate levels should be in good adjustment, but in any case it is advisable to use the more precise level attached to the alidade of the vertical circle, or the telescope level, in plumbing the vertical axis. The effect of the error arising from imperfect leveling may be shewn as follows:

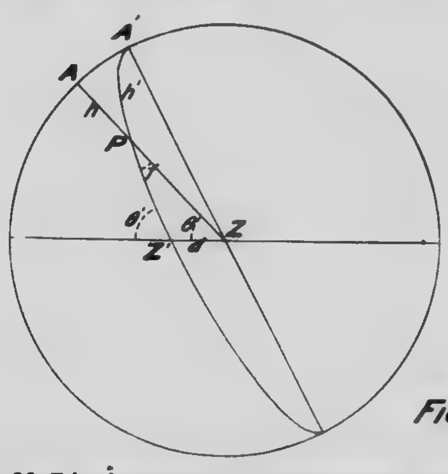


FIG. 29

In Fig. 29 Z is the zenith, Z' the point to which the vertical axis is directed. P is any point. The triangle PZZ' gives the equation

$$\sin \theta' \cot \theta = \sin d \tan h' - \cos d \cos \theta'$$

Then expanding sin d and cos d and neglecting all but the first power of d we have

$$\sin\theta'\cot\theta = d\tan h' + \cos\theta'$$

or

$$\sin\theta'\cos\theta - \cos\theta'\sin\theta = d\tan h'\sin\theta$$

or

$$\sin(\theta'-\theta) = d\tan h'\sin\theta$$

or as $\theta'-\theta$ is small

$$\theta'-\theta = d\tan h'\sin\theta$$

Now if there are two points sighted in turn, and θ_1' and θ_2' are the values which θ' takes, respectively, we have

$$\theta_1'-\theta_1 = d\tan h_1'\sin\theta_1$$

$$\theta_2'-\theta_2 = d\tan h_2'\sin\theta_2$$

so that, taking the difference

$$(\theta_2'-\theta_1')-(\theta_2-\theta_1) = d(\tan h_2'\sin\theta_2 - \tan h_1'\sin\theta_1) \qquad (82)$$

This expresses the error in the horizontal angle between the two points. It appears to be a maximum when $\theta_2 = 270°$ and $\theta_1 = 90°$, and for high altitudes its value may exceed d. It is not eliminated by reversal.

To find the effect on a vertical angle, we have in the triangle APA'

$$\cos f = \frac{\tan h}{\tan h'},$$

or

$$\tan h' = \frac{\tan h}{\cos f} = \frac{\tan h}{1-\frac{f^2}{2}} = \tan h\left(1+\frac{f^2}{2}\right),$$

nearly. Then writing $h' = h + \Delta h$ we have

$$\tan h' = \tan(h+\Delta h)$$

$$= \tan h + \Delta h \sec^2 h$$

by Taylor's theorem. $\therefore$

$$\Delta h \sec^2 h = \frac{f^2}{2}\tan h$$

or

$$\Delta h = \frac{f^2}{2}\tan h\cos^2 h$$

Again, in the triangle PZZ'

$$\sin f = \frac{\sin\theta'\sin d}{\cos h}$$

or

$$f = \frac{d\sin\theta'}{\cos h}$$

Substituting in the above expression for Δh we have

$$\Delta h = \frac{d^2}{2}\,\frac{\sin^2\theta'}{\cos^2 h}\tan h\cos^2 h$$

$$= \frac{d^2}{2} \sin^2 \theta' \tan h$$

$$\text{or in seconds } \Delta h = \frac{\sin 1''}{2} d^2 \sin^2 \theta' \tan h \qquad (83)$$

This is never appreciable.

(5) It is convenient that adjustment 9 be nearly perfect, though not essential, as the effect of imperfect adjustment is eliminated by reversal.

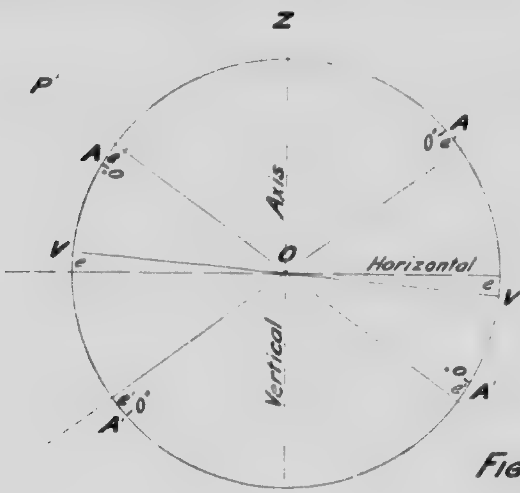


FIG. 30

In Fig. 30 the circle represents the vertical circle of the transit; *OP* is the sight line, directed to some point *P*. The error of *VA*, the reading of the vernier *V*, is evidently =

$$e+e'.$$

If the telescope now be transited, turned in azimuth, and again directed to the point *P*, it amounts to the same thing as transiting and directing to a second point *P'* which has the same absolute zenith distance as *P*. The reverse reading is then *VA'* whose error is =

$$-(e+e')$$

The mean of *VA* and *VA'*, the two readings of vernier *V*, is therefore the altitude of *P* freed from the effect of index error.

To observe an altitude of a heavenly body with a transit.

It has been shewn that errors of adjustment have no appreciable effect upon a vertical angle, except the index

error, whose effect may be eliminated by reversal. In observing the altitude of a star, therefore, the method is to make two pointings to the star, reversing the instrument between the pointings. The telescope is first directed so that the star is very near and approaching the horizontal thread at a point a little to the right or left of the centre. The time of crossing the thread is then noted, and also the V.C.R. The instrument is then reversed and directed as before, with the star at about the same distance on the opposite side of the centre, thus eliminating the effect of any inclination of the thread. The time of passage across the thread is again noted, and the V.C.R. If azimuth is required as well as time, the star must be observed on the intersection of the horizontal and vertical threads. The mean of the two V.C.R's. is then the observed altitude—freed from the effect of index error—corresponding to the mean of the observed times.

It is thus assumed that the change of altitude of a star, during short intervals of time, is proportional to the time. This assumption will seldom lead to an error exceeding $0^s.1$ for an interval of 3^m between the observations.

In observing the sun the same general method is followed as in observing a star, but as there is no definite point at the sun's centre that can be observed, the procedure is as illustrated in Fig. 31. The sun's image is first brought to the

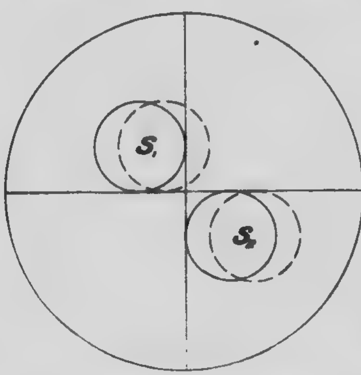


FIG 31

position shewn by the broken circle S_1, so as to be in contact with the horizontal thread and slightly overlapping the vertical thread. It may then be kept in contact with the horizontal thread by turning the altitude tangent screw; its own motion will then bring it into contact with the vertical thread, as shewn by the full circle S_1. After noting and

recording the time and the readings of the circles the instrument is reversed and the observation repeated, bringing the sun into the position S_2. The figure represents an afternoon observation for time and azimuth, taken with an inverting telescope. If time alone is required the contact of the sun's image with the vertical thread is not important. The means of the readings of the two circles may now be regarded as corresponding to a pointing to the sun's centre at an instant equal to the mean of the times.

A form of record is shewn on p. 43.

The Sextant.

The principle and construction of the instrument.

In Fig. 32 AB is the graduated arc, M_1 the index mirror, M_2 the horizon mirror, M_1V the index arm to which the mirror M_1 is attached, and carrying the vernier V at its extremity. The instrument embodies the principle that if a ray of light SM_1 be incident upon the mirror M_1, then

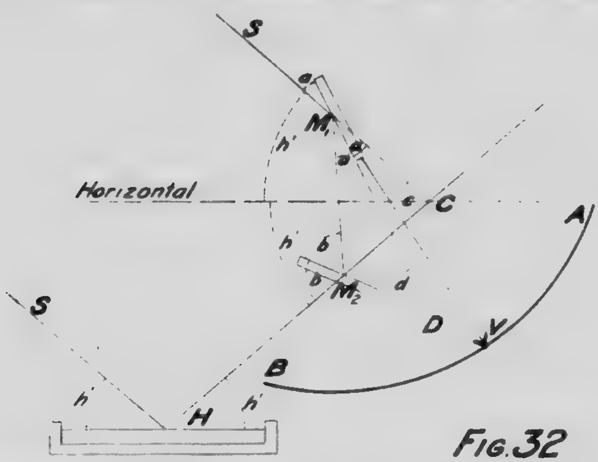


FIG. 32

reflected from it to the mirror M_2, from which it is again reflected, then the angle c between the first and last directions of the ray is equal to double the angle d between the mirrors. This is readily proved, for in the triangles M_1M_2C and M_1M_2D we have, respectively

$$2b = 2a + c$$

and $$b = a + d$$

or $$2b = 2a + 2d$$

$\therefore$ $$c = 2d$$

The mirror M_2 is attached permanently to the frame of the instrument, and half of its surface is unsilvered, while

M_1 is attached to the index arm and turns with it. The sighting telescope is directed along the line CM_2. The mirrors are so placed that when their planes are parallel the index V is at the zero A of the graduated arc AB. The arc is divided into twice the number of degrees that it subtends at its centre M_1.

To measure the angle between two points the instrument is held so that its plane passes through the two points, and the left-hand point is seen in the field of the observing telescope through the unsilvered half of the mirror M_2. The index arm is then turned until the other point, seen by double reflection from the two mirrors, appears to coincide with the first. The reading of the arc is the angle subtended by the two points at the point C. It is to be remarked that C is not a fixed point for all angles.

Adjustment of the sextant.

To observe an altitude of the sun with a sextant and artificial horizon.

The artificial horizon is a horizontal reflecting surface, usually the surface of mercury contained in an iron trough. In observing the altitude of a heavenly body the angle is measured between its image, seen by reflection in the artificial horizon, and that seen by reflection from the mirrors of the instrument. Fig. 32 shews that this angle is equal to double the apparent altitude of the body. In observing the sun, instead of superposing the two images seen in the field of the telescope, it is best to bring them into external contact, thus observing either the upper or the lower limb. As the horizon image appears erect in the field of an inverting telescope, and the other image inverted, the identification of either image shews which limb has been observed.

To determine the index error of the instrument after observing the sun, set the vernier nearly at zero and then direct the sight line to the sun; the two images will now be seen nearly in coincidence. Then turn the tangent screw until the images are in external contact, and read the arc. Then reverse the motion of the screw, causing the images to pass one over the other until they are again in contact, and again read the arc. One of the readings will be on the extra arc. Half the difference of the two readings is the index error, positive if the reading on the extra arc is the greater. The sum of the readings is twice the sun's angular diameter.

8. Formulæ of Spherical Trigonometry.

$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos A\\ \cos b &= \cos a \cos c + \sin a \sin c \cos B\\ \cos c &= \cos a \cos b + \sin a \sin b \cos C\end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned}\cos A &= -\cos B \cos C + \sin B \sin C \cos a\\ \cos B &= -\cos A \cos C + \sin A \sin C \cos b\\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c\end{aligned}\right\} \qquad (2)$$

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \qquad (3)$$

$$\left.\begin{aligned}\sin a \cos B &= \sin c \cos b - \cos c \sin b \cos A\\ \sin a \cos C &= \sin b \cos c - \cos b \sin c \cos A\\ \sin b \cos A &= \sin c \cos a - \cos c \sin a \cos B\\ \sin b \cos C &= \sin a \cos c - \cos a \sin c \cos B\\ \sin c \cos A &= \sin b \cos a - \cos b \sin a \cos C\\ \sin c \cos B &= \sin a \cos b - \cos a \sin b \cos C\end{aligned}\right\} \qquad (4)$$

$$\left.\begin{aligned}\sin A \cot B &= \sin c \cot b - \cos c \cos A\\ \sin B \cot A &= \sin c \cot a - \cos c \cos B\\ \sin B \cot C &= \sin a \cot c - \cos a \cos B\\ \sin C \cot B &= \sin a \cot b - \cos a \cos C\\ \sin A \cot C &= \sin b \cot c - \cos b \cos A\\ \sin C \cot A &= \sin b \cot a - \cos b \cos C\end{aligned}\right\} \qquad (5)$$

$$\left.\begin{aligned}\sin^2 \tfrac{1}{2}A &= \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}\\ \sin^2 \tfrac{1}{2}B &= \frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}\\ \sin^2 \tfrac{1}{2}C &= \frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}\end{aligned}\right\} \qquad (6)$$

where
$$s = \frac{a+b+c}{2}$$

$$\left.\begin{aligned}\cos^2 \tfrac{1}{2}A &= \frac{\sin s \sin(s-a)}{\sin b \sin c}\\ \cos^2 \tfrac{1}{2}B &= \frac{\sin s \sin(s-b)}{\sin a \sin c}\\ \cos^2 \tfrac{1}{2}C &= \frac{\sin s \sin(s-c)}{\sin a \sin b}\end{aligned}\right\} \qquad (7)$$

$$\left.\begin{aligned}\tan^2 \tfrac{1}{2}A &= \frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}\\ \tan^2 \tfrac{1}{2}B &= \frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}\\ \tan^2 \tfrac{1}{2}C &= \frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}\end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned}\sin^2 \tfrac{1}{2}a &= -\frac{\cos S\cos(S-A)}{\sin B\sin C}\\ \sin^2 \tfrac{1}{2}b &= -\frac{\cos S\cos(S-B)}{\sin A\sin C}\\ \sin^2 \tfrac{1}{2}c &= -\frac{\cos S\cos(S-C)}{\sin A\sin B}\end{aligned}\right\} \tag{9}$$

where $$S = \frac{A+B+C}{2}$$

$$\left.\begin{aligned}\cos^2 \tfrac{1}{2}a &= \frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}\\ \cos^2 \tfrac{1}{2}b &= \frac{\cos(S-A)\cos(S-C)}{\sin A\sin C}\\ \cos^2 \tfrac{1}{2}c &= \frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}\tan^2 \tfrac{1}{2}a &= -\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}\\ \tan^2 \tfrac{1}{2}b &= -\frac{\cos S\cos(S-B)}{\cos(S-A)\cos(S-C)}\\ \tan^2 \tfrac{1}{2}c &= -\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}\end{aligned}\right\} \tag{11}$$

Delambre's analogies—

$$\frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}C} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c} \tag{12}$$

$$\frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c} \tag{13}$$

$$\frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2}C} = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c} \tag{14}$$

$$\frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C} = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c} \tag{15}$$

Napier's analogies—

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}\cot \tfrac{1}{2}C \tag{16}$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}\cot \tfrac{1}{2}C \tag{17}$$

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}\tan \tfrac{1}{2}c \tag{18}$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}\tan \tfrac{1}{2}c \tag{19}$$

Formulæ for right-angled triangles—

$$\cos c = \cos a \cos b \qquad (C = 90^\circ) \tag{20}$$

$$\sin A = \frac{\sin a}{\sin c} \qquad \sin B = \frac{\sin b}{\sin c} \tag{21}$$

$$\cos A = \frac{\tan b}{\tan c} \qquad \cos B = \frac{\tan a}{\tan c} \tag{22}$$

$$\tan A = \frac{\tan a}{\sin b} \qquad \tan B = \frac{\tan b}{\sin a} \tag{23}$$

$$\cos c = \cot A \cot B \tag{24}$$

$$\sin A = \frac{\cos B}{\cos b} \qquad \sin B = \frac{\cos A}{\cos a} \tag{25}$$

Solution of oblique-angled triangles.

Case 1.—Given a, b and c, the three sides
Solution by means of equations (6), (7) or (8).

Case 2.—Given A, B and C, the three angles.
Solution by means of equations (9), (10) or (11).

Case 3.—Given two sides and the included angle, as a, b and C.

1st solution—By means of equations (16), (17) and (3).

2nd solution—By means of equations (1) and (5), or

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$
$$\sin C \cot A = \sin b \cot a - \cos b \cos C$$
$$\sin C \cot B = \sin a \cot b - \cos a \cos C$$

These equations become, when adapted for logarithmic computation

$$\tan \theta = \tan a \cos C \qquad \tan \theta_1 = \tan b \cos C$$

$$\cos c = \frac{\cos a \cos (b-\theta)}{\cos \theta} \qquad \tan B = \frac{\tan C \sin \theta_1}{\sin(a-\theta_1)}$$

$$\tan A = \frac{\tan C \sin \theta}{\sin (b-\theta)}$$

Case 4.—Given two angles and the included side, as A, B and c.

1st solution—By means of equations (18), (19) and (4).

2nd solution—By means of equations (2) and (5), or

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$
$$\sin A \cot B = \sin c \cot b - \cos c \cos A$$
$$\sin B \cot A = \sin c \cot a - \cos c \cos B$$

These equations become

$$\tan \theta_2 = \tan A \cos c \qquad \tan \theta_3 = \tan B \cos c$$

$$\cos C = -\frac{\cos A \cos(B+\theta_2)}{\cos \theta_2} \qquad \tan b = \frac{\tan c \sin \theta_3}{\sin(A+\theta_3)}$$

$$\tan a = \frac{\tan c \sin \theta_2}{\sin(B+\theta_2)}$$

Case 5.—Given two sides and an angle opposite one of them, as a, b and A.

1st solution—By means of equations (3), (16) and (18), or

$$\sin B = \frac{\sin b \sin A}{\sin a}$$

$$\tan \tfrac{1}{2}C = \frac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B)$$

$$\tan \tfrac{1}{2}c = \frac{\cos \tfrac{1}{2}(A+B)}{\cos \tfrac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

2nd solution—By means of equations (3), (5) and (1), or

$$\sin B = \frac{\sin b \ \sin A}{\sin a}$$

$$\sin C \cot A = \sin b \cot a - \cos b \cos C$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

These equations may be thus adapted for log's.

$$\tan \theta_4 = \tan A \cos b$$

$$\sin(C+\theta_4) = \tan b \cot a \sin \theta_4$$

$$\tan \theta_5 = \tan b \cos A$$

$$\cos(c-\theta_5) = \frac{\cos a \cos \theta_5}{\cos b}$$

Case 6.—Given two angles and a side opposite one of them, as A B and a.

1st solution—By means of equations (3), (16) and (18), as in the last case, (3) being written

$$\sin b = \frac{\sin B \ \sin a}{\sin A}$$

2nd solution—By means of equations (3), (2) and (5), or

$$\sin b = \frac{\sin B \ \sin a}{\sin A}$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\sin B \cot A = \sin c \cot a - \cos c \cos B$$

Adapting for log's, we have

$$\tan \theta_6 = \tan B \cos a \qquad \tan \theta_7 = \tan a \cos B$$

$$\cos(C+\theta_6) = -\frac{\cos A \ \cos \theta_6}{\cos B} \qquad \sin(c-\theta_7) = \tan B \cot A \sin \theta_7$$

GEODESY.

1. Figure of the Earth.

In any survey the extent of which is such that the curvature of the earth's surface must be taken into consideration, the figure of the earth may be regarded as that of an oblate spheroid, the elements of a meridian section of which are, as determined by Col. A. R. Clarke, 1866:

Major semi-axis, $a = 20926062$ ft.
Minor semi-axis, $b = 20855121$ ft.

Denoting the eccentricity by e we have

$$e^2 = \frac{a^2 - b^2}{a^2}. \tag{1}$$

The following log's are of frequent use:

$$\log a = 7.3206875$$
$$\log e = \bar{2}.9152513$$
$$\log e^2 = \bar{3}.8305026$$
$$\log (1 - e^2) = \bar{1}.9970504$$
$$\log \frac{e^2}{1 - e^2} = \bar{3}.8334522$$
$$\log \sqrt{\frac{e^2}{1 - e^2}} = \bar{2}.9167261$$

Radii of curvature—Any section of the spheroid by a plane is an ellipse. If the plane contains the normal, or plumb line, at a point, the resulting section is a normal section. Any straight line—so called—traced on the earth's surface is therefore a portion of an elliptic arc; for practical purposes, however, if its length does not exceed 100 miles, it may be regarded as a circular arc whose radius is the radius of curvature of the normal section, of which it is a portion, at its middle point. If the normal section coincides with the meridian an expression for its radius of curvature is

$$\rho_m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}. \tag{2}$$

If the normal section is perpendicular to the meridian its radius of curvature is

$$\rho_n = \frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}}. \tag{3}$$

This is also the length of the normal—AN or BN', Fig. 39—terminated in the minor axis of the spheroid. These are termed the "principal radii of curvature" at a point whose latitude is ϕ. The radius of curvature of a normal section whose azimuth is α may be expressed in terms of these; thus

$$\frac{1}{\rho_\alpha} = \frac{\cos^2 \alpha}{\rho_m} + \frac{\sin^2 \alpha}{\rho_n} \tag{4}$$

or

$$\frac{1}{\rho_\alpha} = \frac{1}{\rho_n}\left(1 + \frac{e^2}{1-e^2}\cos^2\phi\cos^2\alpha\right) \tag{5}$$

By substituting in (2) and (3)

$$\sin\theta = e\sin\phi$$

they become

$$\rho_m = a(1-e^2)\sec^3\theta \tag{6}$$

$$\rho_n = a\sec\theta \tag{7}$$

Eq. (4) may also be placed in a convenient form for computation. Thus writing it

$$\rho_\alpha = \frac{\rho_m\,\rho_n}{\rho_n\cos^2\alpha + \rho_m\sin^2\alpha}$$

it may be thus transformed

$$\rho_\alpha = \frac{\rho_n}{\sin^2\alpha + \frac{\rho_n}{\rho_m}\cos^2\alpha} = \frac{\rho_n}{\sin^2\alpha\left(1 + \frac{\rho_n}{\rho_m}\cot^2\alpha\right)}$$

Then writing $\frac{\rho_n}{\rho_m}\cot^2\alpha = \cot^2 x$

it becomes

$$\rho_\alpha = \frac{\rho_n}{\sin^2\alpha(1+\cot^2 x)} = \frac{\rho_n}{\sin^2\alpha\,\mathrm{cosec}^2 x}$$

$$= \rho_n\,\frac{\sin^2 x}{\sin^2\alpha}.$$

ρ_α is then given by the equations

$$\tan x = \sqrt{\frac{\rho_m}{\rho_n}}\tan\alpha \qquad \rho_\alpha = \rho_n\frac{\sin^2 x}{\sin^2\alpha} \tag{8), (9}$$

By expansion in series the log's of these radii of curvature may be thus expressed:

$$\log\rho_m = 7.3199482 - [\bar{3}.3448221]\cos 2\phi + [\bar{6}.27371..]\cos 4\phi - \tag{10}$$

$$\log\rho_n = 7.3214243 - [\bar{4}.8677005]\cos 2\phi + [\bar{7}.79659..]\cos 4\phi - \tag{11}$$

$$\log\rho_\alpha = \log\rho_n - [\bar{3}.4712365]\cos^2\phi\cos^2\alpha + [\bar{5}.00366..]\cos^4\phi\cos^4\alpha - \tag{12}$$

The numbers in brackets are the log's of constant numerical coefficients.

For tables giving the values of ρ_m ρ_n, etc., see the Supplement to the Manual of Dom. Land Surveys.

2. A Trigonometric Survey.

Objects of such a survey.

Choice of stations. Well-conditioned triangles. The base net.

Height of stations in order to overcome the earth's curvature:

Fig. 33

Let A and B be two stations whose heights above sea level are H_1 and H_2, and distance apart s. O is the centre of curvature of the arc s. The curved line AB is the path of the ray of light between the two stations. z is the zenith distance of B observed at A. We have then in the triangle ABO:

$$\frac{BO}{AO} = \frac{\sin BAO}{\sin ABO}$$

or $$\frac{\rho+H_2}{\rho+H_1} = \frac{\sin(z+r)}{\sin(z+r-\sigma)} = \frac{\sin(z+r)}{\sin(z+r)\cos\sigma - \cos(z+r)\sin\sigma};$$

or $$\frac{1+\dfrac{H_2}{\rho}}{1+\dfrac{H_1}{\rho}} = \frac{1}{1-\dfrac{\sigma^2}{2}-\sigma\cot(z+r)};$$

or $$\left(1+\frac{H_2}{\rho}\right)\left(1-\frac{H_1}{\rho}\right) = 1+\sigma\cot(z+r)+\frac{\sigma^2}{2};$$

or $$\frac{H_2-H_1}{\rho}=\sigma(\cot z-r\operatorname{cosec}^2 z)+\frac{\sigma^2}{2};$$

expanding by Taylor's theorem. Then as

$$\rho\sigma=s,\ r=m\sigma,$$

m denoting the coefficient of refraction, and z is nearly 90°, we have

$$H_2-H_1=s\left(\cot z-\frac{ms}{\rho}\right)+\frac{s^2}{2\rho}$$

$$=s\cot z+\frac{s^2}{2\rho}(1-2m) \qquad (13)$$

If H' now be the height of the ray AB at a distance s' from A, we have

$$H'-H_1=s'\cot z+\frac{s'^2}{2\rho}(1-2m).$$

Writing k for $\frac{1-2m}{2\rho}$ and eliminating $\cot z$ between this eq. and (13), we find

$$\frac{H_2-H_1}{s}-\frac{H'-H_1}{s'}=k(s'-s)$$

Then solving for H_1 we have

$$H_1=\frac{H's-H_2s'}{s-s'}+k\,s\,s' \qquad (14)$$

This gives the height necessary for a station at A in order that a distant station B, of known height H_2, may be visible over an intervening elevation H'.

If we solve for H' we have

$$H'=\frac{H_2-H_1}{s}s'+H_1-ks'(s-s'); \qquad (15)$$

which gives the height of the ray of light at a given distance from A.

Clarke gives the following values for m:

For rays crossing the sea, $m=.0809$
For rays not crossing the sea, $m=.0750$

Measurement of a base line—Geodetic base lines are now measured with tapes or wires of invar, an alloy composed of iron and nickel in the proportion of 64 to 36. This material has an extremely small coefficient of expansion, so that the difficulty experienced in determining the temperature correction, when other materials are used, is thus obviated. Good

results may also be obtained with a well standardized steel tape by working in cloudy weather or at night so as to avoid sudden changes of temperature.

In making a measurement the tape is stretched clear of the ground by applying a considerable tension, and rests at its zero points on supports in the form of tripods or stakes driven firmly into the ground. The rear zero division of the tape having been placed in coincidence with a fine mark on the head of its support, the relative positions of the forward zero division of the tape and the mark on its support may then be measured with a scale. The distance between the marks on the two supports may be found by applying certain corrections to the tape length. These corrections are:

For temperature,
For tension,
For sag, and
For grade.

Correction for temperature:

$$c_1 = aL(t - t_o) \qquad (16)$$

in which

L = the standard length of tape;
t_o = the temperature at which it is standard;
t = temperature at time of measurement;
a = coefficient of expansion.

Correction for tension:

$$c_2 = eTL \qquad (17)$$

in which

e = extension of unit length due to unit tension.
T = tension in lbs.

Correction for sag:

$$c_3 = \frac{L^3 w^2}{24\,T^2} = \frac{L}{24}\left(\frac{W}{T}\right)^2 \qquad (18)$$

in which

w = wt. of unit of length of tape
W = wt. of tape.

Correction for grade: Denoting the difference of elevation of the end supports, determined by levelling, by h, we have

$$\begin{aligned} c_4 &= L - (L^2 - h^2)^{\frac{1}{2}} \\ &= L - L\left(1 - \frac{h^2}{L^2}\right)^{\frac{1}{2}} \\ &= L - L\left(1 - \frac{1}{2}\frac{h^2}{L^2} - \frac{1}{8}\frac{h^4}{L^4} - \frac{1}{16}\frac{h^6}{L^6} - \right) \\ &= \frac{1}{2}\frac{h^2}{L} + \frac{1}{8}\frac{h^4}{L^3} + \end{aligned}$$

6

$$= \frac{h}{2} \cdot \frac{h}{L} + \frac{h}{8}\left(\frac{h}{L}\right)^3 + \qquad (19)$$

This first term in this expression is nearly always sufficient.

The following may be used as the coefficients of expansion for steel and invar tapes:

Steel, 0.0000114
Invar, 0.00000041

In the absence of experinental data the extension of a steel tape may be computed from its modulus of elasticity. 28000000 lbs. The extension of invar may be taken to be

0.00000004394 ft.

per lb., per foot, per sq. in. of cross section.

The distance between the supports, reduced to the horizontal, then is

$$L_0 = L + c_1 + c_2 - c_3 - c_4 \qquad (20)$$

Reduction of a base measurement to sea level—

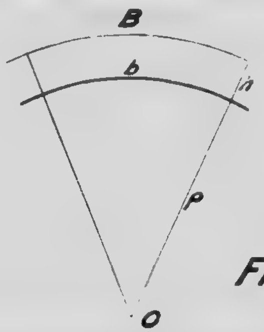


FIG. 34

Let, B = measured length of base, h being its height above sea level;
b = its length reduced to sea level.

Then we have

$$\frac{b}{B} = \frac{\rho}{\rho + h} \qquad \text{or} \quad b = B\frac{\rho}{\rho + h}$$

$$\therefore \qquad B - b = B\left(1 - \frac{\rho}{\rho + h}\right) = B\frac{h}{\rho + h}$$

$$= B\frac{h}{\rho\left(1 + \frac{h}{\rho}\right)} = B\frac{h}{\rho}\left(1 + \frac{h}{\rho}\right)^{-1}$$

$$= B\frac{h}{\rho}\left(1 - \frac{h}{\rho} + \frac{h^2}{\rho^2} - \frac{h^3}{\rho^3} + \right)$$

$$=B\left(\frac{h}{\rho}-\frac{h^2}{\rho^2}+\right). \qquad (21)$$

The first term here is usually sufficient.

A broken base—It is sometimes necessary to measure a base line in two parts, deflecting through a small angle at this point of junction.

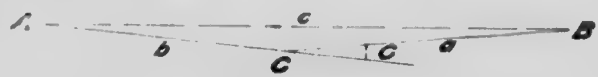


FIG. 35

Let a and b, Fig. 35, be the two parts, making the small angle C with one another. It is required to find the length c We have

$$\begin{aligned} c^2 &= a^2+b^2+2\,ab\cos C,\\ &= a^2+b^2+2ab\left(1-\frac{C^2}{2}\right),\text{ nearly},\\ &= (a+b)^2-abC^2,\\ &= (a+b)^2\left(1-\frac{abC^2}{(a+b)^2}\right),\\ \therefore\quad c &= (a+b)\left(1-\frac{abC^2}{(a+b)^2}\right)^{\frac{1}{2}},\\ &= (a+b)\left(1-\tfrac{1}{2}\frac{abC^2}{(a+b)^2}\right),\text{ nearly}.\\ &= a+b-\tfrac{1}{2}\frac{abC^2}{a+b}, \end{aligned}$$

or, if C is in seconds

$$c=a+b-\frac{\sin^2 1''}{2}\cdot\frac{abC^2}{a+b} \qquad (22)$$

$$\log\frac{\sin^2 1''}{2}=11.0701198$$

To interpolate a portion of a base—Sometimes a portion of a base cannot be directly measured. In Fig. 36, a and b and the angles P Q and R are measured; it is required to find the length x. We have

$$\frac{BE}{a}=\frac{\sin A}{\sin P} \qquad \frac{CE}{a+x}=\frac{\sin A}{\sin Q}$$

$$\therefore\quad \frac{BE}{CE}=\frac{a\sin Q}{(a+x)\sin P}$$

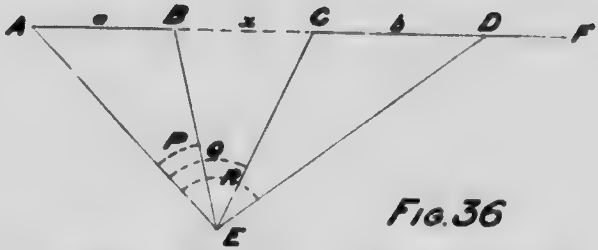


Fig. 36

Again, $\dfrac{BE}{b+x} = \dfrac{\sin(A+R)}{\sin(R-P)} \qquad \dfrac{CE}{b} = \dfrac{\sin(A+R)}{\sin(R-Q)}$

$$\therefore \qquad \frac{BE}{CE} = \frac{(b+x)\ \sin(R-Q)}{b \sin(R-P)}$$

$\therefore$ equating, we have

$$\frac{ab \sin Q \sin(R-P)}{\sin P \sin(R-Q)} = (a+x)\ (b+x)$$

$$= ab + (a+b)x + x^2$$

Then write

$$\tan^2 K = \frac{4ab \sin Q \sin(R-P)}{(a-b)^2 \sin P \sin(R-Q)} \tag{23}$$

and we have

$$x^2 + (a+b)x + ab - \tfrac{1}{4}(a-b)^2 \tan^2 K = 0$$

$$\therefore \qquad x = -\tfrac{1}{2}(a+b) \pm \sqrt{\tfrac{1}{4}(a+b)^2 - ab + \tfrac{1}{4}(a-b)^2 \tan^2 K}$$

$$= -\tfrac{1}{2}(a+b) \pm \sqrt{\tfrac{1}{4}(a-b)^2 + \tfrac{1}{4}(a-b)^2 \tan^2 K}$$

$$= -\tfrac{1}{2}(a+b) \pm \tfrac{1}{2}(a-b) \sec K \tag{24}$$

If $a=b$ this solution fails. In that case write

$$\tan^2 K' = \frac{ab \sin Q \sin(R-P)}{\sin P \sin(R-Q)} \tag{25}$$

then we have

$$x^2 + (a+b)x + ab - \tan^2 K' = 0$$

and $\qquad x = -\tfrac{1}{2}(a+b) \pm \sqrt{\tfrac{1}{4}(a+b)^2 - ab + \tan^2 K'}$

$$= -\tfrac{1}{2}(a+b) \pm \sqrt{\tfrac{1}{4}(a-b)^2 + \tan^2 K'}$$

$$= -\tfrac{1}{2}(a+b) \pm \tan K' \tag{26}$$

Measurement of angles—The angles of a triangulation may be measured either with a direction theodolite, or one of the repetition pattern. The circle of the former instrument is usually read by three equidistant verniers or microscopes. In measuring the angles at a station each of the distant stations is sighted in order, from left to right, and the microscopes read. The telescope is then transited, or reversed in

the standards, and each station is again sighted, in the order from right to left, and the microscopes again read. A value of each angle is thus obtained from each microscope, and in each position of the instrument, direct and reversed. The mean value of the angle thus obtained is free from the effect of eccentricity and errors of adjustment of the instrument. With three microscopes the effect of reversal is to give, for each station sighted, six readings distributed at equal intervals round the circle, thus minimizing the effect of division errors of the circle. If the construction of the stand permits the circle may now be turned to a new position and the angle measurements repeated, etc., thus further diminishing the effect of division errors.

A repetition theodolite is usually read by verniers, and with this pattern of instrument the repetition principle may be used to advantage. It may be thus described:

Let A (the left-hand station) and B be two stations, the angle between which is to be measured.

Point to A and read verniers. Loosen upper clamp and point to B and read verniers. Then loosen lower clamp and again point to A. Then loosen upper clamp and again point to B, thus obtaining a reading equal to double the angle. This process may be repeated until a final reading is obtained equal to, say, six times the angle between the two stations.

Next loosen the lower clamp, transit the telescope, and point to B. Then loosen upper clamp, turn vernier plate in a clockwise direction, and point to A, thus diminishing the final reading of the first set of repetitions by the amount of the angle between the two stations. Repeat this operation as often as in the first set, thus obtaining a final reading approximating closely to the initial reading.

It is to be noted that in both sets of repetitions the vernier plate is always turned in a clockwise direction; that in the first set the instrument is turned from A to B with the upper clamp loose and the lower clamp tight; and that in the second set these conditions are reversed.

The required angle is now found by taking the mean of the differences between the initial and final readings in the two sets, and dividing by the number of repetitions. This result is largely free from the effect of a drag of the circle by the vernier plate.

Reduction of an observed angle to centre of station—This reduction is necessary when for some reason the centre of a station cannot be occupied by the observer.

In Fig. 37 A is the centre of the station, O the point occupied. The angles O β and γ are measured, and the distance m. The angle A is required. We have

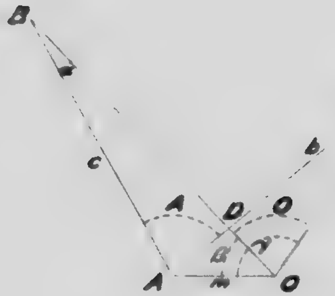


FIG. 37

$$A = BDC - x = O - x + y;$$

and $\qquad \sin x = \dfrac{m \sin \beta}{c} \qquad \sin y = \dfrac{m \sin \gamma}{b}$

Then x and y being small we may substitute their circular measures for their sines, and write them in the form $x \sin 1''$ and $y \sin 1''$, x and y being expressed in seconds, so that we have

$$A = O - \frac{m \sin \beta}{c \sin 1''} + \frac{m \sin \gamma}{b \sin 1''} \tag{27}$$

Distant stations are rendered visible by means of acetylene lamps for night work, and heliotropes for day work. Description of some forms of heliotrope.

3. Computation of the Triangulation.

The portion of the surface of the spheroid contained within a triangle is assumed to be a portion of a spherical surface whose radius is the geometric mean of the principal radii of curvature at the central point of the triangle.

Spherical excess of a triangle—It is shewn in spherical geometry that the sum of the angles of a spherical triangle exceeds two right angles by an amount termed the "spherical excess" of the triangle.

To find the spherical excess of a given triangle:

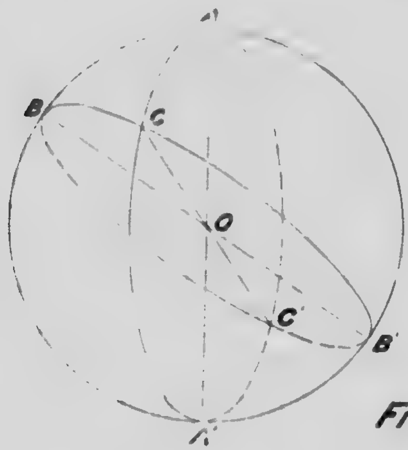


Fig. 38

Let ABC be a spherical triangle, and $A'B'$ and C' points diametrically opposite A B and C. The surface of the hemisphere is made up of the three lunes $ABA'C$, $BCB'A$, and $CAC'B$ this last being equal to the sum of the two triangles CAB and $CA'B'$—less twice the area of the triangle ABC. Denoting these by Lune A, etc., and the area of the triangle by Δ, we have

$$\text{Lune } A = \frac{A}{\pi} 2\pi R^2 = 2AR^2$$

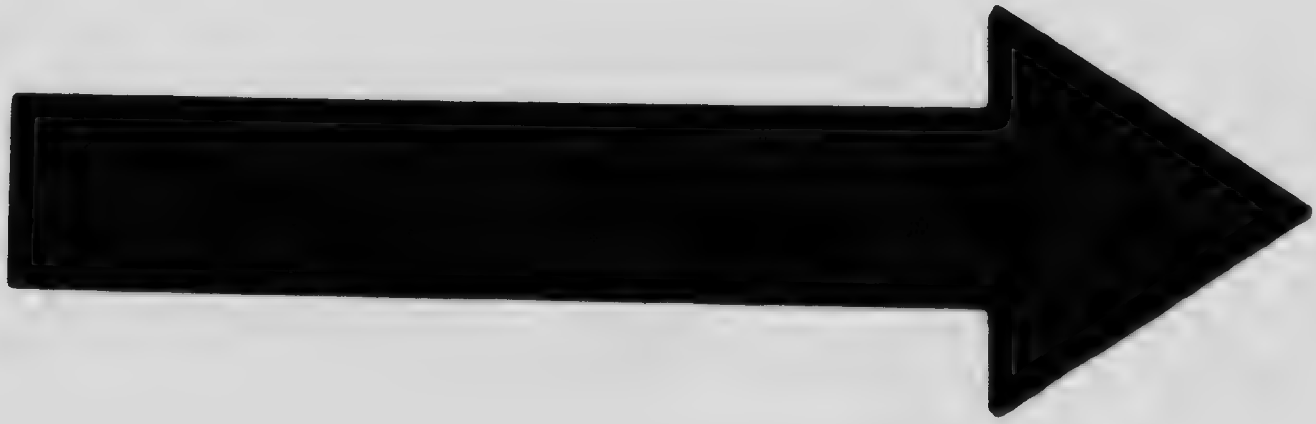
$$\text{Lune } B = 2BR^2$$

$$\text{Lune } C = 2CR^2$$

$$\therefore \quad 2AR^2 + 2BR^2 + 2CR^2 - 2\Delta = 2\pi R^2$$

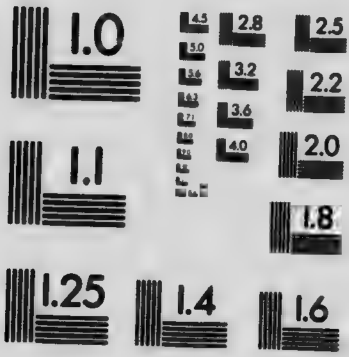
or

$$A + B + C - \pi = \frac{\Delta}{R^2}$$



MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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or, denoting the spherical excess by ϵ we have in seconds

$$\epsilon = \frac{\Delta}{R^2 \sin 1''} \tag{28}$$

For a triangle on the earth's surface this may be written

$$\epsilon = \frac{\Delta}{\rho_m \rho_n \sin 1''} \tag{29}$$

The area of the triangle, in all but extreme cases, may be computed as if the triangle were plane, so that we may write

$$\epsilon = \frac{ab \sin C}{2\rho_m \rho_n \sin 1''} \tag{30}$$

or

$$\epsilon = \frac{a^2 \sin B \sin C}{2\rho_m \rho_n \sin 1'' \sin (B+C)} \tag{31}$$

The value of $1/2 \rho_m \rho_n \sin 1''$—which we may denote by m—may be computed by the expression

$$\log \frac{1}{2\rho_m \rho_n \sin 1''} = \overline{10}.372023 + [\bar{3}.469754] \cos 2\phi \tag{32}$$

the number in brackets being the log. of a constant coefficient.

The following table was computed by (32):

ϕ	m	ϕ	m
40°	$\overline{10}$.37253	50	$\overline{10}$.37151
41	243	51	141
42	233	52	131
43	223	53	121
44	213	54	111
45	202	55	101
46	192	56	092
47	182	57	082
48	171	58	073
49	161	59	064
		60	055

Legendre's theorem—This theorem may be thus stated: If the sides of a spherical triangle are small in comparson with the radius of the sphere, it may be solved as a plane triangle by first diminishing each angle by one-third of the spherical excess of the triangle.

To prove this, let

A B and C be the angles of the triangle,

a b and c the sides, expressed in radians,

$A'B'$ and C' the angles of a plane triangle, whose sides α β and γ have the same lengths expressed in feet as those of the spherical triangle.

Then we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{1-\dfrac{\alpha^2}{2r^2}+\dfrac{\alpha^4}{24r^4}-\left(1-\dfrac{\beta^2}{2r^2}+\dfrac{\beta^4}{24r^4}\right)\left(1-\dfrac{\gamma^2}{2r^2}+\dfrac{\gamma^4}{24r^4}\right)}{\left(\dfrac{\beta}{r}-\dfrac{\beta^3}{6r^3}\right)\left(\dfrac{\gamma}{r}-\dfrac{\gamma^3}{6r^3}\right)}$$

$$= \frac{1-\dfrac{\alpha^2}{2r^2}+\dfrac{\alpha^4}{24r^4}-\left(1-\dfrac{\beta^2}{2r^2}+\dfrac{\beta^4}{24r^4}-\dfrac{\gamma^2}{2r^2}+\dfrac{\beta^2\gamma^2}{4r^4}+\dfrac{\gamma^4}{24r^4}\right)}{\dfrac{\beta\gamma}{r^2}-\dfrac{\beta\gamma^3}{6r^4}-\dfrac{\beta^3\gamma}{6r^4}}$$

$$= \frac{\dfrac{\beta^2+\gamma^2-\alpha^2}{2r^2}+\dfrac{\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2}{24r^4}}{\dfrac{\beta\gamma}{r^2}\left(1-\dfrac{\beta^2+\gamma^2}{6r^2}\right)}$$

$$= \left(\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2}{24\beta\gamma r^2}\right)\left(1+\frac{\beta^2+\gamma^2}{6r^2}\right)$$

$$= \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2}{24\beta\gamma r^2}$$

$$+\frac{\beta^4+\beta^2\gamma^2-\alpha^2\beta^2+\beta^2\gamma^2+\gamma^4-\alpha^2\gamma^2}{12\beta\gamma r^2}$$

$$= \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4+\beta^4+\gamma^4-2\alpha^2\beta^2-2\alpha^2\gamma^2-2\beta^2\gamma^2}{24\beta\gamma r^2} \qquad (a)$$

Now in the triangle $A'B'C'$ we have

$$\cos A' = \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma} \qquad (b)$$

$$\therefore \quad \sin^2 A' = 1-\left(\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}\right)^2$$

$$= -\frac{\alpha^4+\beta^4+\gamma^4-2\alpha^2\beta^2-2\alpha^2\gamma^2-2\beta^2\gamma^2}{4\beta^2\gamma^2} \qquad (c)$$

$\therefore$ by (a) (b) and (c) we have

$$\cos A = \cos A' - \sin^2 A' \frac{\beta\gamma}{6r^2} \qquad (d)$$

Then assume $\quad A = A'+\theta$
and we have $\cos A = \cos A' - \theta \sin A'$
by Taylor's theorem. Therefore comparing with (d) we have

$$\theta \sin A' = \sin^2 A' \frac{\beta\gamma}{6r^2}$$

or
$$\theta = \frac{\beta\gamma \sin A'}{6r^2} = \frac{1}{3r^2} \cdot \tfrac{1}{2}\beta\gamma \sin A'$$
$$= \frac{\Delta}{3r^2} = \frac{\epsilon}{3}$$

This proves the theorem.

If the three angles of a triangle are measured, the spherical excess may be computed by (30) or (31) using the values of the angles given by measurement. The closing error then is

$$180° + \epsilon - (A + B + C)$$

which may be divided among the angles, giving to each a correction which is inversely proportional to its weight. One third of the spherical excess is then deducted from each angle, and the triangle solved as a plane triangle. If the three angles have equal weights the closing error may therefore be found as if the triangle were plane and divided equally among them.

For triangles the lengths of whose sides do not greatly exceed 6 miles the error due to the neglect of spherical excess is not likely to amount to 0.01 ft.

In the case of a triangulation consisting of an intricate chain or network of triangles, the angles must be subjected to a rigid process of adjustment before the triangles are solved. The adjustment of a triangulation constitutes a subject in itself, which is beyond the scope of these notes. (Leading principles outlined).

4. Geodetic Positions.

The latitude and longitude of one of the stations, and the azimuth of a triangle side extending from that station, having been determined astronomically, the geographical co-ordinates of all the stations of the triangulation may now be computed. The problem thus presented for solution is:

Given the latitude and longitude of a point on the earth's surface, and the length and initial azimuth of the line drawn from it to a second point, to determine the latitude and longitude of this point, and the azimuth of the first point as seen from the second.

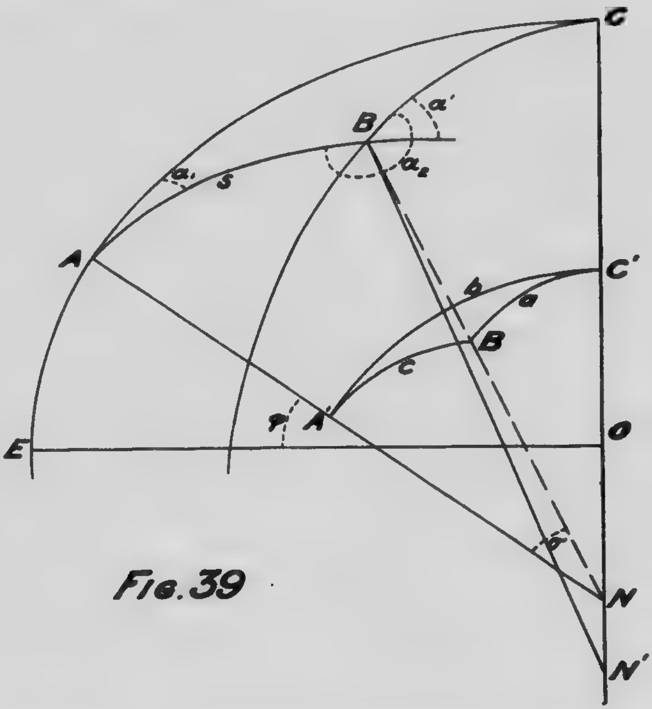

Fig. 39

In Fig. 39 A is the first point and B the second; C is the ... AC and BC are the meridians of A and B. O is the ...re of the spheroid. AN and BN' are normals to the

spheroid at the points A and B. $A'B'C'$ is a spherical triangle, the centre of the sphere being at N. We have given then

$$\phi_1 \ \alpha_1 \text{ and } s$$

and are required to find

$$\phi_2 \ \Delta L \text{ and } \alpha_2$$

To find $\Delta\phi(=\phi_2-\phi_1)$—

In the triangle $A'B'C'$ we have given b c and $A'(=\alpha_1)$, and must find $a(=90°-\phi_2')$, $C(=\Delta L)$, and B.

We have

$$\cos a=\cos b \cos c+\sin b \sin c \cos A'$$

or

$$\sin \phi_2'=\sin \phi_1 \cos c+\cos \phi_1 \sin c \cos \alpha_1$$

$$=\sin \phi_1\left(1-\frac{c^2}{2}\right)+c \cos \phi_1 \cos \alpha_1$$

or

$$\sin \phi_2'-\sin \phi_1=c \cos \phi_1 \cos \alpha_1-\frac{c^2}{2}\sin \phi_1$$

But

$$\sin \phi_2'-\sin \phi_1=\sin(\phi_1+\Delta\phi')-\sin \phi_1$$

$$=\sin \phi_1\left(1-\frac{\Delta\phi'^2}{2}\right)+\Delta\phi' \cos \phi_1-\sin \phi_1$$

$$=\Delta\phi' \cos \phi_1-\frac{\Delta\phi'^2}{2}\sin \phi_1$$

$$\therefore \quad \Delta\phi' \cos \phi_1-\frac{\Delta\phi'^2}{2}\sin \phi_1=c \cos \phi_1 \cos \alpha_1-\frac{c^2}{2}\sin \phi_1$$

or

$$\Delta\phi'-\frac{\Delta\phi'^2}{2}\tan \phi_1=c \cos \alpha_1-\frac{c^2}{2}\tan \phi_1.$$

Assuming as a first approximation

$$\Delta\phi'=c \cos \alpha_1$$

and substituting in the term in $\Delta\phi'^2$, we have

$$\Delta\phi'=c \cos \alpha_1-\frac{c^2}{2}\tan \phi_1+\frac{c^2}{2}\tan \phi_1 \cos^2 \alpha_1$$

$$=c \cos \alpha_1-\frac{c^2}{2}\tan \phi_1 \sin^2 \alpha_1 \qquad (33)$$

Then substituting $c=\dfrac{s}{N}$

we have ($\Delta\phi'$ being in seconds)

$$\Delta\phi'=\frac{s \cos \alpha_1}{N \sin 1''}-\frac{1}{2}\left(\frac{s \cos \alpha_1}{N \sin 1''}\right)^2 \tan \phi_1 \tan^2 \alpha_1 \sin 1'' \qquad (34)$$

This gives the difference of latitude on an imaginary sphere whose radius is $N(=\rho_n)$, whereas the radius should be

assumed equal to the value of ρ_m for the mean of the latitudes of A and B, or, with sufficient precision, for the latitude $\phi_1+\frac{1}{2}\Delta\phi'$. We have then

$$\Delta\phi=\Delta\phi'\ \frac{N}{\rho_m} \tag{35}$$

Also $$\phi_2=\phi_1+\Delta\phi \tag{36}$$

To find ΔL—

Again, in the triangle $A'B'C'$, we have

$$\sin C' = \frac{\sin c \sin A'}{\sin a}$$

or $$\sin \Delta L = \frac{\sin c\ \sin \alpha_1}{\cos \phi_2'}$$

or, substituting arcs for sines

$$\Delta L = \frac{c \sin \alpha_1}{\cos \phi_2'}$$

or in seconds $$\Delta L = \frac{s \sin \alpha_1}{N \sin 1'' \cos \phi_2'} \tag{37}$$

To find $\Delta\alpha(=\alpha'-\alpha_1)$—

We have

$$\tan \tfrac{1}{2}(A'+B') = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C'$$

But
$$\begin{aligned} A'+B' &= \alpha_1+180^\circ-\alpha', \\ &=180^\circ-(\alpha'-\alpha_1), \\ &=180^\circ-\Delta\alpha; \\ a-b &=90^\circ-\phi_2-90^\circ+\phi_1, \\ &=-(\phi_2-\phi_1)=-\Delta\phi; \\ a+b &= 90^\circ-\phi_2+90^\circ-\phi_1 \\ &= 180^\circ-(\phi_1+\phi_2); \end{aligned}$$

$\therefore$ $$\cot \tfrac{1}{2}\Delta\alpha = \frac{\cos \frac{1}{2} \Delta\phi}{\sin \phi_m} \cot \tfrac{1}{2} \Delta L;$$

or $$\tan \tfrac{1}{2}\Delta\alpha = \frac{\sin \phi_m}{\cos \frac{1}{2}\Delta\phi} \tan \tfrac{1}{2} \Delta L;$$

or, substituting arcs for tangents

$$\Delta\alpha=\Delta L\ \frac{\sin \phi_m}{\cos \frac{1}{2}\Delta\phi}\ . \tag{38}$$

This is termed the convergence of the meridians of A and B. Then finally

$$\begin{aligned} \alpha_2 &=180^\circ+\alpha' \\ &=180^\circ+\alpha_1+\Delta\alpha \end{aligned} \tag{39}$$

An expression giving $\Delta\alpha$ directly in terms of the data is sometimes useful. It may be derived as follows: Taking the equation

$$\sin A' \cot B' = \sin c \cot b - \cos c \cos A',$$

it may be thus transformed

$$\tan B' = \frac{\sin \alpha_1}{\sin c \cot b - \cos c \cos \alpha_1}$$

$$= -\frac{\sin \alpha_1}{\cos \alpha_1 \left(\cos c - \sin c \dfrac{\cot b}{\cos \alpha_1}\right)}$$

$$= -\frac{\tan \alpha_1}{1 - \dfrac{c^2}{2} - c \dfrac{\cot b}{\cos \alpha_1}}$$

$$= -\tan \alpha_1 \left(1 + \frac{c \cot b}{\cos \alpha_1} + \frac{c^2}{2} + \frac{c^2 \cot^2 b}{\cos^2 \alpha_1}\right)$$

$$\therefore \quad \tan B' + \tan \alpha_1 = -\tan \alpha_1 \left(\frac{c \cot b}{\cos \alpha_1} + \frac{c^2}{2} + \frac{c^2 \cot^2 b}{\cos^2 \alpha_1}\right)$$

But $B = 180° - \alpha'$, $\therefore$

$$\tan \alpha' - \tan \alpha_1 = \tan \alpha_1 \left(\frac{c \cot b}{\cos \alpha_1} + \frac{c^2}{2} + \frac{c^2 \cot^2 b}{\cos^2 \alpha_1}\right)$$

Also $\alpha' = \alpha_1 + \Delta\alpha$, $\therefore$ by Taylor's theorem

$$\tan \alpha' = \tan(\alpha_1 + \Delta\alpha)$$
$$= \tan \alpha_1 + \Delta\alpha \sec^2 \alpha_1 + \Delta\alpha^2 \tan \alpha_1 \sec^2 \alpha_1$$

$\therefore$, substituting, we have

$$\Delta\alpha \sec^2 \alpha_1 + \Delta\alpha^2 \tan \alpha_1 \sec^2 \alpha_1 = \tan \alpha_1 \left(\frac{c \cot b}{\cos \alpha_1} + \frac{c^2}{2} + \frac{c^2 \cot^2 b}{\cos^2 \alpha_1}\right)$$

or
$$\Delta\alpha + \Delta\alpha^2 \tan \alpha_1 = c \cot b \sin \alpha_1 + \frac{c^2}{2} \sin \alpha_1 \cos \alpha_1 + c^2 \cot^2 b \tan \alpha_1$$

Assuming as a first approximation

$$\Delta\alpha = c \cot b \sin \alpha_1$$

and substituting in the term containing $\Delta\alpha^2$ we find after reduction

$$\Delta\alpha = c \cot b \sin \alpha_1 + \frac{c^2}{2} \sin \alpha_1 \cos \alpha_1 (1 + 2 \cot^2 b)$$

or in seconds

$$\Delta\alpha = \frac{s}{N} \frac{\tan \phi_1 \sin \alpha_1}{\sin 1''} + \frac{1}{2} \left(\frac{s}{N}\right)^2 \frac{\sin \alpha_1 \cos \alpha_1}{\sin 1''} (1 + 2 \tan^2 \phi_1)$$

By writing (40)

$$x = s \sin \alpha_1 \quad y = s \cos \alpha_1$$

equations (34), (35), (37) and (40) become

$$\Delta\phi = \frac{y}{\rho_m \sin 1''} - \frac{x^2 \tan \phi_1}{2\rho_m \rho_n \sin 1''} \tag{41}$$

$$\Delta L = \frac{x}{\rho_n \cos \phi_2' \sin 1''} \tag{42}$$

$$\Delta\alpha = \frac{x \tan \phi_1}{\rho_m \sin 1''} + \frac{x\,y}{2\rho_n^2 \sin 1''} (1 + 2 \tan^2 \phi_1) \tag{43}$$

These equations should not be used for distances exceeding 20 miles. (38) should be used in preference to (40) or (43) when all the unknown quantities are required.

For longer distances—approaching 100 miles—the following equations may be used:

$$x = \frac{s \sin \alpha_1}{\rho_n} \qquad y = \frac{s \cos \alpha_1}{\rho_n}$$

$$\Delta\phi' = \frac{y}{\sin 1''} + \frac{y^3 \tan^2 \alpha_1}{3 \sin 1''} - \frac{x^2 \tan \phi'}{2 \sin 1''} \tag{44}$$

$$\phi' = \phi_1 + \text{1st two terms}$$

$$\Delta\phi = \Delta\phi' \frac{\rho_n}{\rho_m}$$

$$\Delta L = \frac{x}{\cos \phi_2' \sin 1''} + \frac{\sin^2 1''}{6} (\Delta L')^3 \left(1 - \frac{\cos^2 \phi_2'}{\sin^2 \alpha_1}\right) \tag{45}$$

$$\Delta L' = \text{1st term} \qquad \phi_2' = \phi_1 + \Delta\phi'$$

$$\Delta\alpha = \frac{\Delta L \sin \phi_m}{\cos \frac{1}{2}\Delta\phi} - \frac{\sin^2 1''}{12} (\Delta\alpha')^3 \left(1 - \frac{\cos^2 \frac{1}{2} \Delta\phi}{\sin^2 \phi_m}\right) \tag{46}$$

$$\phi_m = \phi_1 + \tfrac{1}{2}\Delta\phi \qquad \Delta\alpha' = \text{1st term.}$$

The following log's are here useful:

1/ sin 1″ = 5.31442513	log sin² 1″/6 = $\overline{12}$.59300
1/3 sin 1″ = 4.83730	log sin² 1″/12 = $\overline{12}$.29197
1/2 sin 1″ = 5.0133951	

ple.—Let $s = 20$ miles, $\phi_1 = 44° 30'$, $\alpha_1 = 48° 20'$.

To find $\Delta\phi'$, eq. (34)—

log s (in ft.)	= 5.0236639
log cos α_1	= 9.8226883
log ρ_n	= $\overline{7}$.3214108
log sin 1″	= $\overline{6}$.6855749

	4.8463522
	2.0069857
log 690.8225	$=$ 2.8393665
	5.67873
log 0.5	$=\bar{1}.69897$
log tan ϕ_1	$=$ 9.99242
log $\tan^2 \alpha_1$	$=$ 10.10129
log sin 1″	$=\bar{6}.68557$
log 1.4355	$=$ 0.15698
$\Delta\phi'$	$=689''.387$
	$=11'\ 29''.387$

To find $\Delta\phi$, eq. (35)—

log $\Delta\phi'$	$=$ 2.8384631
log ρ_n	$=$ 7.3214108
log ρ_m	$=$ 7.3199151
	10.1598739
log $\Delta\phi$	$=$ 2.8399588
$\Delta\phi$	$=691''.765$
	$=11'\ 31''.765$
ϕ_1	$=44^\circ\ 30'$
ϕ_2	$=44^\circ\ 41'\ 31''.765$

To find ΔL, eq. (37)—

log s	$=$ 5.0236639
log sin α_1	$=$ 9.8733352
log ρ_n	$=$ 7.3214108
log sin 1″	$=\bar{6}.6855749$
log cos ϕ_2'	$=$ 9.8518109
	4.8969991
	1.8587966
log 1091.952	$=$ 3.0382035
ΔL	$=1091''.952$
	$=18'\ 11''.952$

The second term in eq. (45) in this example $=0''.0005$.

To find $\Delta\alpha$, eq. (38)—

$\log \Delta L$	$= 3.0382075$
$\log \sin \phi_m$	$= 9.8464016$
$\log \cos \frac{1}{2}\Delta\phi$	$= 9.9999993$
	2.8846091
log 766.672	$= 2.8846098$
$\Delta\alpha$	$= 12' \, 46''.672$

The second term in eq. (46, here amounts to $0''.001$.

To find α_2, eq. (39)—

α_1	$= 48° \, 20' \, 00''$
$\Delta\alpha$	$= 12' \, 46''.672$
	$180° \, 00' \, 00''$
α_2	$= 228° \, 32' \, 46''.672$

The above equations (41), (42), (43) and (38) may readily be adapted for the solution of a variety of problems. Thus—

given ϕ_1 ϕ_2 and ΔL

to find α_1 α_2 and s.

We have
$$x = \Delta L \,.\, \rho_n \cos \phi_2 \sin 1'' \tag{47}$$

$$y = \Delta\phi \,.\, \rho_m \sin 1'' + \tfrac{1}{2} \frac{x^2 \tan \phi_1}{\rho_m \, \rho_n \sin 1''} \rho_m \sin 1''$$

$$= \Delta\phi \,.\, \rho_m \sin 1'' + \frac{x^2 \tan \phi_1}{2\rho_n} \tag{48}$$

Then
$$\tan \alpha_1 = \frac{x}{y} \tag{49}$$

$$\Delta\alpha = \Delta L \frac{\sin \phi_m}{\cos \frac{1}{2} \Delta\phi}$$

$$\alpha_2 = 180° + \alpha_1 + \Delta\alpha$$

$$s = \frac{x}{\sin \alpha_1} = \frac{y}{\cos \alpha_1} \tag{50}$$

Again, given ϕ_1 ad α_1,

to find s ΔL and α_2.

We have from (48) and (49)

$$y = \Delta\phi \,.\, \rho_m \sin 1'' + \frac{y^2 \tan^2 \alpha_1 \tan \phi_1}{2\rho_n}$$

$$= \Delta\phi \,.\, \rho_m \sin 1'' + (\Delta\phi \,.\, \rho_m \sin 1'')^2 \frac{\tan^2 \alpha_1 \tan \phi_1}{2\rho_n}$$

$$x = y \tan \alpha_1 \tag{51}$$

$$s = \frac{x}{\sin \alpha_1} = \frac{y}{\cos \alpha_1}$$

$$\Delta L = \frac{x}{\rho_n \cos \phi_2 \sin 1''}$$

Any other problem in which three of these six quantities are given may be solved in a similar manner.

The foregoing equations may be used in reducing to differences of latitude and longitude the courses of a traverse line. Only the first terms are here necessary, so that we may write

$$x = s \sin \alpha \quad y = s \cos \alpha$$

$$\Delta \phi = \frac{y}{\rho_m \sin 1''}$$

$$\Delta L = \frac{x}{\rho_n \cos \phi \sin 1''}$$

$$\Delta \alpha = \frac{x \tan \phi}{\rho_n \sin 1''} = \Delta L \sin \phi \qquad (52)$$

In latitude 45° the maximum values of the second terms of the above expressions, for a length of 1 mile, are, respectively

0''.0066

.0093

.0098

The use to be made of $\Delta \alpha$ is to correct the azimuth of a course referred to the meridian of the initial station of the traverse, to refer it to the meridian of the initial point of the course. As a correction it is additive. The algebraic signs of x and y must be carefully observed.

5. Certain Problems which occur in the Dominion Lands System of Survey.

A general description of that system of survey.

(1) To find the amplitude of a meridian arc having a given length; and conversely.

We have

$$\Delta\phi = \frac{s}{\rho_m \sin 1''} \tag{53}$$

$\Delta\phi$ being in seconds; and conversely

$$s = \Delta\phi \cdot \rho_m \sin 1'' \tag{54}$$

If the arc is at a height H above sea level, then

$$\Delta\phi = \frac{s}{(\rho_m + H)\sin 1''}$$

$$= \frac{s}{\rho_m\left(1 + \dfrac{H}{\rho_m}\right)\sin 1''}$$

$$= \frac{s}{\rho_m \sin 1''}\left(1 - \frac{H}{\rho_m}\right) \tag{55}$$

nearly. Conversely

$$s = \Delta\phi \cdot \rho_m \sin 1''\left(1 + \frac{H}{\rho_m}\right) \tag{56}$$

Example.—Find the amplitude of an arc whose length is 24 miles, middle latitude 52°, and height above sea level 1200 feet.

Eq. (55)	log 24	$= 1.3802112$
	log 5280	$= 3.7226339$
	log s (in ft.)	$= 5.1028451$
	log ρ_m	$= 7.3204817$
	log sin 1''	$= \bar{6}.6855749$
		2.0060566
	log 1249.650	$= 3.0967885$
	log H	$= 3.07918$
	log ρ_m	$= 7.32048$
		6.17597
	log 0.0717	$= \bar{2}.85549$
	$\Delta\phi$	$= 1249.578$
		$= 20' \; 49''.578$

For finding the length of a meridian arc exceeding about a degree the following expression may be used:

$$\begin{aligned} s = & [5.56182842]\Delta\phi \text{ (in degrees)} \\ & -[5.0269884] \cos 2\phi_0 \sin \Delta\phi \\ & +[2.0527848] \cos 4\phi_0 \sin 2\Delta\phi \\ & -[\bar{1}.17356..] \cos 6\phi_0 \sin 3\Delta\phi + \end{aligned}$$

in which

$\Delta\phi$ = the difference of latitude of its extremities,

ϕ_0 = the mean of the extreme latitudes.

The numbers in brackets are logarithms.

This expression is sufficient for finding the length of a whole quadrant.

(2) Given two points on the same parallel of latitude, at a given distance apart, to find their difference of longitude, and the convergence of their meridians.

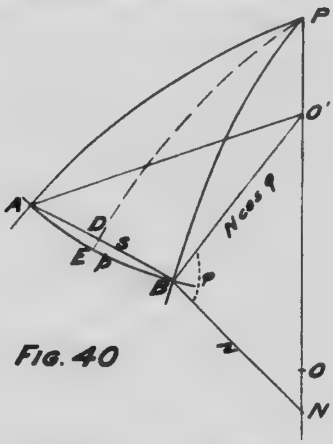

FIG. 40

A and B are the two points; ADB a normal section, and AEB a parallel of latitude. PD is drawn at right angles to ADB. The triangle PDB gives

$$\sin BPD = \frac{\sin BD}{\sin PB}$$

or

$$\sin \frac{\Delta L}{2} = \frac{\sin \frac{s}{2N}}{\cos \phi}$$

or, as ΔL is assumed to be small, this may be written

$$\Delta L = \frac{s}{N \cos \phi}$$

or in seconds
$$\Delta L = \frac{s}{N \cos\phi \sin 1''} \tag{58}$$

If the higher powers of ΔL and $s/2N$ are retained in the expansions, this becomes

$$\Delta L = \frac{s}{N\cos\phi \sin 1''} + \frac{\sin^2 1''}{24} (\Delta L')^3 \sin^2\phi \tag{59}$$

in which $\Delta L'$ is the first term. As

$$N\cos\phi = P,$$

the radius of the parallel of latitude, this may be written

$$\Delta L = \frac{s}{P\sin 1''} + \frac{\sin^2 1''}{24}\left(\frac{s}{P\sin 1''}\right)^3 \sin^2\phi \tag{60}$$

For a chord 6 miles in length, in latitude 52°, the second term of (60) amounts to only 0''.00008, a quantity quite inappreciable, so that the first term may be considered exact.

Again, in the triangle PDB we have

$$\cos PBD = \frac{\tan BD}{\tan PB} \tag{61}$$

or
$$\sin\frac{\Delta\alpha}{2} = \frac{\tan\frac{s}{2N}}{\cot\phi}$$

or
$$\Delta\alpha = \frac{s\tan\phi}{N}$$

—$\Delta\alpha$ being small—; or in seconds

$$\Delta\alpha = \frac{s\tan\phi}{N\sin 1''} \tag{62}$$

The higher terms are here also inappreciable. From (58) and (61) we have

$$\Delta\alpha = \Delta L \sin\phi$$

(See eq. 52).

The deflection angle between two consecutive chords of the same length is clearly

$$\Delta\alpha = \frac{s\tan\phi}{N\sin 1''}$$

and the azimuth of a chord at either extremity

$$90° - \frac{\Delta\alpha}{2}$$

To find the difference in length of s and the arc of the parallel p we have

$$\Delta L = \frac{s}{N\cos\phi} + \frac{1}{24}\left(\frac{s}{N\cos\phi}\right)^3 \sin^2\phi,$$

and $\Delta L = \dfrac{p}{N\cos\phi}$;

Equating these we have

$$p-s=\frac{1}{24}\left(\frac{s}{N\cos\phi}\right)^3\sin^2\phi\, N\cos\phi$$

$$=\frac{s}{24}\left(\frac{s}{N}\right)^2\tan^2\phi \qquad (63)$$

To find the length of an offset from the chord to the parallel of latitude.

Applying eq. (33) to the arc DE, Fig. 40, we have, denoting AD and DE by x and y, respectively,

$$\frac{y}{N}=\frac{x}{N}\cos\alpha-\frac{1}{2}\left(\frac{x}{N}\right)^2\tan\phi\sin^2\alpha$$

and by (61) $\qquad \cos\alpha=\frac{s}{2N}\tan\phi$

$\therefore$ writing $\sin^2\alpha=1$ we have

$$\frac{y}{N}=\frac{x}{N}\cdot\frac{s}{2N}\tan\phi-\frac{x}{2N^2}\tan\phi$$

$$=\frac{x(s-x)}{2N^2}\tan\phi$$

or $$y=\frac{x(s-x)}{2N}\tan\phi. \qquad (64)$$

6. Trigonometric Levelling.

A and B are two stations whose difference of elevation is to be determined; A' and B' are the apparent positions of A and B, affected by refraction. The altitude h of B, observed at A, and the distance s, are assumed to be known.

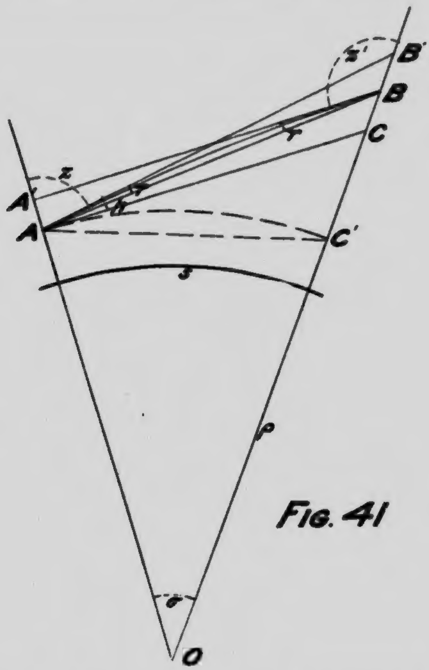

Fig. 41

Denoting the height BC' of B above A by H, we have

$$H = AC' \frac{\sin BAC'}{\sin ABC'}$$

But $$BAC' = h - r + CAC' = h - r + \frac{\sigma}{2},$$

$$= h - m\sigma + \frac{\sigma}{2},$$

$$= h + (\tfrac{1}{2} - m)\sigma;$$

and $$ABC' = 90° - h + r - \sigma$$

$$= 90° - h + m\sigma - \sigma$$

$$= 90° - \{h + (1-m)\sigma\}.$$

$$\therefore \qquad H = s\,\frac{\sin\left\{h+(\tfrac{1}{2}-m)\sigma\right\}}{\cos\left\{h+(1-m)\sigma\right\}}. \qquad (65)$$

See Supp. to Manual of Dominion Land Surveys.

For the numerical value of m see p. 76.

In eq. (65) it is assumed that the distance s is equal to the chord AC'. If A and B are stations of a trigonometric survey and s is obtained by the solution of a triangle, then it is the distance AB reduced to sea level. The correction to s for elevation is

$$s\,\frac{H_1}{\rho},$$

H_1 being the height of A above sea level. Also the correction to reduce from the arc to the chord is

$$\frac{s'}{24}\left(\frac{s}{\rho}\right)^2,$$

so that the length of the chord AC' is

$$s\left(1+\frac{H_1}{\rho}\right)\left\{1-\frac{1}{24}\left(\frac{s}{\rho}\right)^2\right\},$$

the second correction only becoming appreciable for considerable distances.

Reciprocal zenith distances—

If the zenith distances z and z' be observed simultaneously at the two stations the effect of refraction is eliminated, if it can be assumed to affect the two zenith distances equally. Thus, returning to the above equation for H, we have

$$BAC' = 90° - z - r + \frac{\sigma}{2}$$

$$ABC' = 180° - z' - r$$

But we have also

$$A'AB = z + r = 180° - (z'+r) + \sigma$$

so that

$$r = \frac{180° - z - z' + \sigma}{2}$$

which therefore becomes known. Substituting this we have

$$BAC' = \frac{z'-z}{2}$$

$$ABC' = 90° - \frac{z'-z+\sigma}{2}$$

$\therefore$ substituting in the first above expression for H gives

$$H = s\,\frac{\sin\tfrac{1}{2}(z'-z)}{\cos\tfrac{1}{2}(z'-z+\sigma)} \qquad (66)$$

s having been corrected for elevation, and if necessary for curvature.